Quantitative Methods for Management

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DIRECTOR

QUANTITATIVE METHODS FOR MANAGEMENT

CONTENTS

Notes

UNIT 1 PERMUTATION AND COMBINATION

Structure

1.0 OBJECTIVES

After going through this unit you will be able to understand:

- The meaning and definition of permutation.
- The meaning and concept of combination.
- The Matrix definition, types and operations like addition, subtraction
- The Determinants – definitions, properties and it's problems.

1.1 INTRODUCTION

The process of selecting things is called combination and that of arranging the selected things is called permutations.

Notes

Permutation and combination provide the rules of counting the different numbers in a wide variety of problems relating to statistics and quantitative techniques. More particularly, in a problem relating to probability, the knowledge of permutation and combination is indispensable. An outline of these two concepts is given here as under.

1.2 MEANING OF PERMUTATION

A permutation is an arrangement is a definite order of a number of objects taken some or all at a time. It refers to the maximum possible number of arrangements that can be made of a given number of things taking one or more of them at a time in different possible orders. For example, if there are three things say a, b and c, they can be arranged in the following different possible ways taking one, two or three at a time respectively.

Possible arrangements of a, b, c in different order

Nature of pairs	Possible arrangements	Total number of arrangements
Taking one at a time	a, b, c	3
Taking two at a time	ab, ac, bc, ba, ca, cb	6
Taking all at a time	abc, acb, bca, bac, cab, cba	6

1.3 RULES OF PERMUTATION

From the above analysis, the different rules of permutation may be outlined as follows:

General Rule

The number of permutations of n different things taken r at a time is denoted by ${}^{n}P_{r}$

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

(Sign ! is called factorial)

Where n = number of different things given and n! is read as "n factorial" which implies the product of n (n –1) (n – 2) ! etc.

r = number of tings taken at a time viz : one, two, three etc.

In the above example of p, q, r, the number of arrangements in each of the above pairs can be easily computed by the application of this rule as follows:

(i) Taking one at a time

Here, n = 3, and r = 1

We have, $${}^{n}P_{r} = \frac{n!}{(n-r)!} = {}^{3}P_{1} = \frac{3!}{(3-1)!} = \frac{3 \times 2 \times 1}{2 \times 1} = 3$$

(ii) Taking two at a time

Here, n = 3, and r = 2

We have, $${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\therefore \; {}^{3}P_{2} = \frac{3!}{(3-2)!} = \frac{3 \times 2 \times 1}{1} = 6$$

Notes

(iii) Taking all at a time

Here, $n = 3$, and $r = 3$

We have, $${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\therefore {}^{3}P_{3} = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$$

Note that $0! = 1$

Proof:

We have, ${}^{n}P_{n} – n!$ (by arranging the pairs)

Also $${}^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

$$\therefore n! = \frac{n!}{0!} \text{(by substituting } {}^{n}P_{n} \text{ by n!)}$$

$$\therefore 0! = \frac{n!}{n!} = 1$$

Thus the results shown above confirms with the results shown earlier by way of practical presentations of the different arrangement.

THEOREM - I (Permutation with Repetitions)

The number of permutations of n different things taken r at a time when each thing may be repeated r times is

$$\boxed{n(P) = n^{r}}$$

THEOREM - II (Permutation of n things not all different)

The number of n things taken all at a time of which p things are alike, q things are alike, r things are alike, and the rest all are different is given by

$$\boxed{n(p) = \frac{n!}{p!q!r!}}$$

Thus the word 'combination' which contains two 'o' s, two 'i' s and two 'n' s, and five letters of dissimilarity can have the following number of permutations which is given by

$$\boxed{n(p) = \frac{n!}{p!q!r!}}$$

Where, $n = 11$

$P = 2$ (it represents 'o' s)

$q = 2$ (it represents 'i' s)

And $r = 2$ (it represents 'n' s)

Thus, $$n(p) = \frac{11!}{2!2!2!} = 5034960$$

Notes

THEOREM - III (Rule of Counting)

If an operation can take place in m ways, and the same having taken place in one of these ways, a second operation can take place in n ways, the number of ways in which the two operations can take place is given by

$$\boxed{n(p) = m \times n}$$

Example: There are 6 different routes from Bhubaneswar to Delhi. In how many ways can a person go from Bhubaneswar to Delhi by one route and return by another?

Solution.

According to the proposition, a man can go to Delhi in 6 ways but can return to Bhubaneswar in 5 ways only, as he can not return in the route through which he went.

Thus, the total number of ways of performing the journey is given by

$$n(p) = m \times n$$

$$= 6 \times 5 = 30 \text{ ways}$$

Note that the above rule can be extended to cases involving more than two operations, where

$$n(p) = m \times n \times \ldots \text{ etc.}$$

Special Rules

(i) The number of ways in which all n things can be arranged in order is given by

$$n(p) = n!$$

$$= n(n-1)(n-2)\ldots(n-n+1)$$

Example: Find the number of ways in which 5 encyclopedias can be arranged in a shelf.

Solution.

The required number of ways is given by

$$n(p) = n!$$

$$= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

(ii) The number of ways in which (p + q +r) things can be divided into three groups containing p, q and r things respectively is given by

$$n(p) = \frac{(p+q+r)!}{p!q!r!}$$

Example: In how many ways can 6 books be arranged in three shelves containing 3, 2, and 1 books respectively?

Solution:

The possible number of ways of arranging the 6 books in the three shelves in groups of 3, 2 and 1 is given by

$$n(p) = \frac{(p+q+r)!}{p!\,q!\,r!} = \frac{(3+2+1)!}{3!\,2!\,1!}$$

$$\frac{6\,!}{3!\,2!\,1!} = 60$$

3. Rules of Circular Permutation

(a) The number of circular arrangements of n things is given by

$$\boxed{n(p) = (n-1)!}$$

Example: Ascertain the number of ways in which 7 persons can be seated in a circular manner.

Solution:

The required number of ways is given by

$$n(p) = (7-1)!$$

$$= 6! = 720$$

(b) The number of circular arrangements of n things of which p are alike and q are alike taken alternatively is given by

$$\boxed{n(p) = (p-1)!\,q!}$$

Example: In how many ways 5 men and 5 women can be seated around a table so that no 2 men are adjacent.

Solution:

Let p = men, q = women

Thus, p = 5 q = 5

The number of ways in which 5 men and 5 women can be seated alternatively in a circle is given by

$$n(p) = (p-1)! \times q!$$

$$= (5-1)! \times 5!$$

$$= 4! \times 5! = 2880$$

(c) The number of circular arrangements of n different things such that no two similar things are adjacent is given by

$$\boxed{n(p) = \frac{1}{2}(n-1)!}$$

Example: In how many ways can 5 persons be seated around a table so that none of them is adjacent to his neighbour?

Solution:

The required number of ways is given by

$$n(p) = \frac{1}{2}(n-1)!$$

$$= \frac{1}{2}(5-1)!$$

$$= \frac{1}{2} \times 4! = \frac{4 \times 3 \times 2 \times 1}{2} = 12$$

Notes

4. Restricted Permutations

(i) The number of permutations of n things taken r at a time in which p particular things do not occur is obtained by

$$n(p) = {}^{n-p}Pr$$

(ii) The number of permutations of n different things taken r at a time in which p particular things are present is given by

$$n(p) = {}^{n-p}P_{r-p} \times {}^{r}P_{p}$$

Illustrations on Permutation

Illustration 1: Ascertain the number of permutations that can be made of the four digits 1, 2, 3 and 4 taking (i) 3 at a time and (ii) all at a time.

Solution:

(i) Taking 3 at a time

Here, n = 4 and r = 3

∴ The required number of permutation is given by

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$= {}^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24$$

(ii) Taking all at a time

Here, n = 4 and r = 4

∴ The required number of permutation is given by

$${}^{n}P_{r} = {}^{4}P_{4} = \frac{4!}{(4-4)!} = \frac{4!}{0!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{1} = 24$$

Thus, the 4 digits 1, 2, 3, and 4 can be arranged in 24 ways when both 3 and all are taken at a time.

Illustration 2: Five persons appear in a musical test in which there are two prizes to be awarded to the persons securing the first and the second positions. Determine the number of ways in which the two prizes may be awarded.

Solution:

Here, number of persons, or n = 5

Number of prizes or r = 2

Thus the number of ways in which the prizes may be awarded is given by

$$n(p) = {}^{5}P_{2} = \frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

= 20

Illustration 3: After publication of a supplementary result, 4 students have applied for 5 hostels. In how many ways can they be accommodated in the hostels?

Solution:

Here, $n = 5$ and $r = 4$

And $${}^{n}p_{r}\,{}^{5}P_{4} = \frac{5!}{(5-4)!} = \frac{5\times4\times3\times2\times1}{1}$$

$= 120$

Thus, the 4 students can be accommodated in the 5 hostels in 120 different ways.

Illustration 4: Find the number of ways in which the kings of a pack of playing cards can be arranged in a row.

Solution:

Here, number of the king cards, or $n = 4$ and $r = 4$

We have, $$n(P) = {}^{n}P_{r} = {}^{4}P_{4} = \frac{4!}{(4-4)!} = \frac{4!}{0!}$$

$$= \frac{4\times3\times2\times1}{1} = 24$$

Thus, the 4 kings can be arranged in 24 ways in a row.

Illustration 5: In how many ways can 7 students be accommodated in 3 hostels?

Solution:

Here, $n = 7$ and $r = 3$

We have, $$n(P) = {}^{n}P_{r} = {}^{7}P_{3} = \frac{7!}{(7-3)!} = \frac{7!}{4!}$$

$= 7 \times 6 \times 5 = 210$ ways

Illustration 6: Find the number of permutations that can be made from the word, "amalgamation".

Solution:

Here, total number of letters in the word, "amalgamation" or $n = 12$, number of the letter 'a' in the word, or $p = 4$, and number of the letter 'm' in the word or $q = 2$.

The required number of permutations when p things are alike, and q things are alike, and the rest are dissimilar is given by.

$$n(p) = \frac{n!}{p!q!} = \frac{12!}{4!2!} = 9979200$$

1.4 COMBINATION

Combination refers to the maximum possible number of arrangement that can be made of a given number of things taking one, or more of them at a time in only one of the different possible orders. In other words, it refers to the number of different sets, or groups that can be made of a given lot taking one, or more at a time without repeating an element. For example, if there are three things say a, b and c, the following are the different possible combinations which can be formed of them taking one, two or three at a time respectively.

Notes

Possible Combinations from abc

Nature of pairs	Combinations	Number of arrangements
Taking one at a time	a, b, c	3
Taking two at a time	ab, ac, bc	3
Taking all at a time	abc	1

Rules of Combination

From the above analysis, the rule of combination may be outlined here as follows:

The number of combinations of n different things taken r at a time is denoted by n_{cr}

$$^{n}C_{r} = \frac{n\,!}{(n-1)!r!}$$

Where, n = number of different things given

r = number of things taken at a time

n! = n factorial which runs like n (n – 1) (n – 2) !

And r ! = r factorial which runs like r (r – 1) (r – 2) !

In the above example of a, b and c things, the number of combinations in each of the above cases can be readily computed by the application of this rule as follows:

(1) ***Taking one at a time:***

Here, n = 3 and r = 1

$\therefore$ Number of combinations, or $^{n}C_{r} = \frac{n\,!}{(n-r)\,r!}$

$$\Rightarrow \quad ^{3}C_{1} = \frac{3\,!}{(3-1)\,!1!} = \frac{3!}{2!\ \ 1\,!} = 3$$

(2) ***Taking two at a time:***

Here n = 3, and r = 2

$\therefore$ Number of combinations, or $^{3}C_{2} = \frac{3\,!}{(3-2)\ !\ 2\ !} = \frac{3 \times 2 \times 1}{1 \times 2 \times 1} = 3$

(3) ***Taking two at a time:***

Here, n = 3 and r = 3

$\therefore$ Number of combinations, or $^{n}C_{r} = {}^{3}C_{3} = \frac{3\,!}{(3-3)\,!\,3!} = \frac{3!}{0!3!}$

$$= \frac{3 \times 2 \times 1}{1 \times 3 \times 2 \times 1} = 1$$

Note that 0 ! = 1

Thus, the results shown above confirm with the result shown earlier by way of practical presentation of the different combinations.

Notes

From the above rule, the inter-relation between permutation and Combination may be noted as follows:

When Permutation = $^nP_r = \frac{n!}{(n-r)!}$

Combination = $^nC_r = \frac{n!}{(n-r)!r!}$

Thus $^nP_r = {}^nC_r \times r!$

And $^nC_r = \frac{^nP_r}{r!}$

Example 1: Find the number of permutations from 5 different things taken 2 at a time, and determine the number of combination therefrom.

Solution:

Here, n = 5 and r = 2

Number of permutations or

$$^nP_r = {}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 \times 4 = 20$$

Number of combinations, therefore, is given by

$$\frac{^nP_r}{r!} = \frac{20}{2!} = \frac{20}{2 \times 1} = 10$$

This can be proved as under:

$$^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = 10$$

Example 2: In how many ways can 3 persons be chosen out of 5 persons (i) without repetition and (ii) with repetition.

Solution:

(i) It is a case of finding the number of combinations which is given by

$$^nC_r = \frac{n!}{(n-r)!r!}$$

Where, n = 5 and r = 3

Thus, $^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!\,3!} = \frac{5 \times 4}{2 \times 1} = 10$ ways

(ii) It is a case of finding the number of permutations which is given by

$$^nP_r = {}^nC_r \times r!$$

Where $^nC_r = 10$ and r = 3

Thus $^nP_r = 10 \times 3! = 10 \times 3 \times 2 \times 1 = 60$ ways

This can be proved as follows:

$$^nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(5-3)!}$$

Notes

$$= \frac{5!}{2!} = 5 \times 4 \times 3 = 60 \text{ ways}$$

6. Restricted Combinations

(a) The number of combinations of n things taken r at a time in which p particular things always occur is given by $^{n-p}C_{r-p}$

(b) The number of combinations of *n* things taken *r* at a time in which p particular things never occur is given by $^{n-p}C_r$

(c) The total number of combinations of n different things taken some or all at a time is given by $r^n - 1$.

Illustrations on Combination

Illustration 1: In an examination paper on Quantitative Methods, 12 questions are set. In how many ways can a candidate choose 5 questions out of them?

Solution:

It is a case of combination since, there will be no repetition of a question selected once.

Here, n = 12, and r = 5

Hence, the required number of combinations is given by

$$^{n}C_r = \frac{n!}{(n-r)!r!} = \frac{12!}{(12-5)!5!}$$

$$= \frac{12!}{7!5!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792$$

$\therefore$ The candidate can select 5 questions out of the 12 questions in 792 different ways.

Illustration 2: In how many ways can one commerce professor, and one economic professor be selected from a staff of professors consisting of 12 commerce and 8 economic professors?

Solution:

1 out of 12 Commerce professors can be selected in $^{12}C_1$ ways.

Where, $$^{12}C_r = \frac{12!}{(12-1)!1!} = \frac{12!}{11!1!} = 12 \text{ ways}$$

1 out of 8 Economics Professors can be selected in $^{8}C_1$ ways.

Where, $$^{8}C_1 = \frac{8!}{(8-1)!1!} = \frac{8!}{7!1!} = 8 \text{ ways}$$

Thus, the required number of selection is given by

$$^{12}C_1 \times {}^{8}C_1 = 12 \times 8 = 96 \text{ ways.}$$

Illustration 3: From among 10 boys, and 8 girls, 7 are to be selected for a particular purpose. In how many ways the selection can be made such that there should be exactly 4 boys in the group?

Solution:

In order that there should be exactly 4 boys in the group 3 girls must be selected with them.

Out of 10 boys, 4 boys can be selected in $^{10}C_4$ ways.

Out of 8 girls, 3 girls can be selected in $^{8}C_3$ ways.

Thus, the required number of selection is given by

Notes

$$^{10}C_4 \times {}^{8}C_3 = \frac{10!}{6!4!} \times \frac{8!}{5!3!}$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 11760$$

Illustration 4: For 5 posts of Readers in a University, 20 persons have applied. In how many ways can be selection be made if,

One particular candidate is always included

(ii) One particular candidate is always excluded?

Solution:

(i) When a particular candidate is to be included always the remaining 4 candidates can be selected out of 19 candidates in $^{19}C_4$ ways vide the rule (a) of the restricted combinations, i.e. $n(C) = {}^{n-p}C_{r-p}$

Thus $$n(C) = \frac{19!}{15!4!} = \frac{19 \times 18 \times 17 \times 16}{4 \times 3 \times 2} = 8376 \text{ ways}$$

(ii) When a particular candidate is to be excluded, the choice is restricted to 5 candidates out of the remaining 19 candidates. Now 5 candidates can be selected in $^{19}C_5$ ways vide the rule (b) of the restricted combination i.e. $n(C) = {}^{n-p}C_r$

Thus, $$n(C) = \frac{19!}{14!5!} = \frac{19 \times 18 \times 17 \times 16 \times 15}{5 \times 4 \times 3 \times 2} = 11628 \text{ ways}$$

Notes

DETERMINANTS

1.5 MEANING OF DETERMINANTS

Determinant is a factor which decisively effects the nature and outcome of something. In linear algebra, the determinant is an useful value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted by Δ, det. A or |A|. Determinants were used by Gottried W. Leibniz in the year 1693 and subsequently, Gabriel Cramer devised Cramer's rule for solving Linear systems in 1750. Determinants play an importance role in finding the inverse of a matrix and also solving the system of linear equations.

French Mathematician Augustin-Louis Cauchy (1789-1857) used determinant in its modern form in the year 1812. Cauchy's work is the most complete of the early works on the determinants.

The various important factors of a matrix viz. Adjoint, Inverse, Rank, Consistency, etc. are very much dependent upon the value of the determinant of the given matrix. Thus, a knowledge of determinant system is highly necessary in the solution of many business problems involving matrices and simultaneous equations.

1.6 DEFINITION OF DETERMINANTS

The terms, determinant can be defined as "a numerical value obtained from a square matrix of the coefficients of certain unknown variables enclosed by two bars by the process of diagonal expansion to tell upon a given algebraic system".

Let, $a_{11}\,x + a_{12}y = 0$

$A_{21}\,x + a_{22}y = 0$

By eliminating x and y, we get the expression as

$a_{11}\,a_{22} - a_{21}\,a_{12} = 0$

Now, we can write the coefficients of the above equation in rows and express in the form of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ which is known as Determinant of Second Order. Thus, the determinant of 2^{nd} order is defined as or 2 × 2 order and the Δ or D may be used as value of the determinant.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$$

Similarly, we can define the determinant of the order three or 3 × 3 order be,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22} - a_{33} - a_{32}a_{23}) - a1_2\,(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}\,(a_{21}\,a_{32} - a_{31}\,a_{22})$$

Example 1: Evaluate:

(i) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & -3 \\ 5 & -4 \end{vmatrix}$ (iii) $\begin{vmatrix} 2x+1 & x^2+x+1 \\ (x-1) & 2x-1 \end{vmatrix}$

Solution:

(i) $\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$

Notes

(ii) $\Delta = \begin{vmatrix} 6 & -3 \\ 5 & -4 \end{vmatrix} = 6 \times (-4) - 5 \times (-3) = -24 + 15 = -9$

(iii) $\Delta = \begin{vmatrix} 2x + 1 & x^2 - x + 1 \\ (x + 1) & 2x - 1 \end{vmatrix} = (2x+1)(2x-1) - (x+1)(x^2 - x + 1)$

$= 4x^2 - 1 - (x^3 + 1) = 4x^2 - x^3 - 2$

Example 2: Evaluate:

(i) $\begin{vmatrix} 3 & 2 & 5 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

Solution:

(i) $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 3 (5 \times 9 - 8 \times 6) - 2(4 \times 9 - 7 \times 6) + 5 (4 \times 8 - 7 \times 5)$

$= 3 (45 - 48) - 2 (36 - 42) + 5 (32 - 35)$

$= 3x (-3) - 2x (- 6) + 5x (- 3)$

$= -9 + 12 - 15 = -12$

(ii) $\Delta = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix} = 3\begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4\begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5\begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix}$

$= 3[2 \times 7 - 1 \times (-3)] - 4 [-6x7 - 8x (-3)] + 5 [-6x1 - 8x2]$

$= 3(14 + 3) - 4 (-42 + 24) + 5 (-6 - 16) = 13$

1.7 CHARACTERISTICS OF DETERMINANTS

From the above definition, the essential characteristics of a determinant may be analysed as under:

(a) **It is a numerical value.** This means that it is expressed in terms of certain numerical figures, or quantity viz 5, 10, 0, -7, -13 etc. For example $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4 - 2 \times 3) = -2$

(b) **It is obtained from a square matrix.** This implies that a determinant can be had only from a square matrix and not from any other matrix whose number of rows and columns are not equal. The square matrix may, however, be of any order viz. 2 × 2 (read as 2 by 2), 3 × 3 (read as 3 by) or of any higher order.

The examples of such matrices are:

$\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}_{2x2}$, $\begin{vmatrix} 3 & 4 & 7 \\ 8 & 9 & 5 \\ 1 & 2 & 9 \end{vmatrix}_{3x3}$,etc.

(c) **It is obtained from the coefficients of certain unknown variables say x, y, z etc.** This means that a determinant is worked out only from the coefficients constituting a system and not from any other elements of the system viz, constants variables, etc. For example, let the

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linear equations relating to a phenomenon be 2x + 3y = 7, and 5x + y = 11. The square matrix of the coefficients for findings the determinants of the above system would be $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}_{2 \times 2}$

(d) **It is obtained from a square matrix enclosed by two bars in its left and right hand sides.** Thus, in the above example, before calculating the value of the determinant the given square matrix is represented as $\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$. This system is adopted only to avoid confusion in distinguishing a determinant from a matrix.

(e) **It has a large number of algebraic properties.** The determinant of a matrix has a large number of algebraic properties for which its value can be determined straight way without undergoing the normal procedure which is usually very lengthy and tedious one.

1.8 COFACTORS AND MINOR OF AN ELEMENT

Cofactor

By the cofactor of an element of a determinant we mean the product of $(-1)^{i+j}$ and the minor of the concerned element (M_{ij}). Symbolically it is given by

$$C_{ij} = (-1)^{i+j} . M_{ij}$$

Where, C_{ij} = the cofactor of the element in the *i*th row and *j*th column of the determinant.

$(-1)^{i+j}$ = the factor determining the algebraic sign depending upon the number of rows (i) and number of columns (j) in which the element occurs in the determinant.

M_{ij} = the minor of the element in the *i*th row and *j*th column of the determinant.

Thus, $C_{11} = (-1)^{1+1} . M_{11}$; $C_{12} = (-1)^{1+2} . M_{12}$, and $C_{13} = (-1)^{1+3} . M_{13}$

Minor

By the minor of an element of a determinant we mean the sub-square-determinant of the given determinant along which the particular element (a_{ij}) does not exit. It is obtained by deleting the row and the column on which the particular element (a_{ij}) lies. It is represented by M_{ij} which denotes the minor of an element in the *i*th row and *j*th column of the determinant. Its value is obtained by deducting the product of its non-leading diagonal elements from the product of its leading diagonal elements.

Example:

Let the determinant $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The minors of its elements in the first row and first column (a_{11}) first row and second column (a_{12}) and in the first row and third column (a_{13}) will be respectively as follows:

The minor of a_{11}, i.e. $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}.a_{33} - a_{32}.a_{23})$

The minor of a_{12} , i.e. $M_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}.a_{33} - a_{31}.a_{23})$

The minor of a_{13}, i.e. $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}.a_{32} - a_{31}.a_{22})$

Notes

From the above analysis it may be noted that, the minors and cofactors are mostly equal except that under certain cases they differ in sign only. We can write the expansion of a determinant in terms of minors and cofactors of the elements, i.e.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3} = a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13}$$

The following examples will show how minors and cofactors of a determinant are determined and numerical value of a determinant is calculated.

Example 3: Find the minors and cofactors of the following:

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}_{2 \times 2}$$

Solution:

Here , $a_{11} = 2$, $a_{12} = 3$, $a_{21} = 1$, and $a_{22} = 5$

(i) ***Minors***

The minor of a_{11} , i.e. $M_{11} = 5$, The minor of a_{12} i.e. $M_{12} = 1$

The minor of a_{21} , -i.e. $M_{21} = 3$, The minor of a_{22} i.e. , M = 2

(ii) ***Cofactors***

The cofactors of an element (C_{ij}) is given by $C_{ij} = (-1)^{i+j} M_{ij}$

The cofactors of a_{11}, i.e. $C_{11} = (-1)^{1+1} M_{11} = 1 \times 5 = 5$;

The cofactors of a_{12}, i.e. $C_{12} = (-1)^{1+2} . M_{12} = -1 \times 1 = -1$;

The cofactors of a_{21}, .e. $C_{21} = (-1)^{2+1} . M_{21} = -1 \times 3 = -3$; and

The cofactors of a_{22}, i.e. $C_{22} = (-1)^{2+2}. M_{22} = 1 \times 2 = 2$

1.9 PROPERTIES OF A DETERMINANT

Determinants have a good number of algebraic properties which help us in finding out the numerical values of the determinants at an ease and straightway. Besides, they also help us in applying the elementary operations over the row and columns of a determinants to simplify its form and arrive at the value at a quicker rate. These properties hold good for the determinants of any order but we shall probe them to the bottom only with the determinants upto the order 3.

Some such important properties are enumerated here as under:

1. If any row or column of the determinant consists of zeros only then the value of a determinant becomes zero.

Proof:

(a) Let the determinant of order 2 be $|A| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}$ and the value of the determinant be Δ.

Then, $\Delta = |A| = (1 \times 0 - 0 \times 2) = 0$ [$\therefore R_1$ is of zeros only]

Notes

(b) Let the determinant of order 3 be, $|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & 6 \end{vmatrix}$ and the value of the determinant be Δ.

Then $\Delta = |B| = 1\begin{vmatrix} 0 & 4 \\ 0 & 6 \end{vmatrix} - 0\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 5 & 0 \end{vmatrix}$

= 1 (0-0) – 0 (18-20) + 2 (0-0)

= 0 – 0 + 0 = 0 [$\therefore C_2$ is of zeros only]

2. **If any two rows or columns of the determinant are identical then the value of the determinant becomes zero.**

Proof:

(a) Let the determinant of order 2 be $|C| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$ and the value of the determinant be Δ.

Thus, Δ = |C| = (1 × 2 – 1 × 2) = 0 [$\therefore R_1$ and R_2 are identical]

(b) Let the determinant of the order 3 be, $|D| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 1 \end{vmatrix}$ and the value of the determinant be Δ.

Expanding the |D| by R_1 we get,

Then, $\Delta = |D| = 1\begin{vmatrix} 5 & 1 \\ 6 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix}$

=1 (5-6) – 2 (1-1) + 1 (6-5)

= - 1 – 0 + 1 = 0 [$\therefore C_1$ & C_3 are identical]

3. **The value of the determinant remains unchanged even if its rows and columns are interchanged.**

Proof:

(a) Let the determinant of the order 2 be, be $|A| = \begin{vmatrix} 4 & 7 \\ 9 & 3 \end{vmatrix}$ and Δ be the value of the determinant and Δ be the value of transposed determinant.

Then Δ = |A| = (4 × 3 – 9 × 7) = (12 – 63) = –51

If the above |A| is transposed then we get

$$|A| = \begin{vmatrix} 4 & 9 \\ 7 & 3 \end{vmatrix}$$

$\therefore$ Δ' = (4 × 3 – 7 × 9) = 12 – 63 = -51

Hence, Δ = Δ'

(b) Let the determinant of the order 3 be $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Notes

Then $\Delta = |B| = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

= 1 (45 – 48) – 2 (36 – 42) + 3 (32 – 35)

= -3 + 12 – 9 = 0

If, the given |B| is transposed then we get,

$$\therefore |B|' = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$$

$\therefore \Delta'$ = 1 (45 – 48) – 4 (18 – 24) + 7 (12 – 15)

= - 3 + 24 – 21 = 0

Hence, $\Delta = \Delta'$

4. If any two adjacent rows or columns of a determinant are interchanged, the numerical value of the determinant remains the same but with the opposite sign.

Proof:

(a) Let the determinant of the order 2 be $|A| = \begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix}$ and the value of the determinant be Δ.

$\Delta = |A| = \begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix} = 5 \text{ x } 4 - 6 \text{ x } 3 = 20 - 18 = 2$

And, Interchanging the Row -1 and Row -2 we get, $\begin{vmatrix} 6 & 4 \\ 5 & 3 \end{vmatrix}$

$\Delta_1 = \begin{vmatrix} 6 & 4 \\ 5 & 3 \end{vmatrix} = (6 \times 3 - 5 \times 4) = (18 - 20) = -2$

And interchanging the Column – 1 and column – 2 we get,

$$\Delta_2 = \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} = (3 \times 6 - 4 \times 5) = (18 - 20) = -2 \quad \therefore \Delta = -\Delta_1 = -\Delta_2$$

(b) Let the determinant of order 3 be $|B| = \begin{vmatrix} 4 & 1 & 5 \\ 6 & 3 & 6 \\ 9 & 2 & 7 \end{vmatrix}$ and the value of the determinant be Δ.

Expanding the above by the row 1 we get, $\Delta|B| = \begin{vmatrix} 4 & 1 & 5 \\ 6 & 3 & 6 \\ 9 & 2 & 7 \end{vmatrix}$ = 4 (21 – 22) – 1 (42-54) + 5 (12 – 27) = 36 + 12 – 75 = -27.

Expanding the above by the Row – 1 we get.

$$\Delta_1 = \begin{vmatrix} 1 & 4 & 5 \\ 3 & 6 & 6 \\ 2 & 9 & 7 \end{vmatrix} = 1(42 - 54) - 4(21 - 12) + 5(27 - 12)$$

= -12 – 36 + 75 = 27

Notes

Again, interchanging the two adjacent rows 2 and 3 we get, $\begin{vmatrix} 4 & 1 & 5 \\ 9 & 2 & 7 \\ 6 & 3 & 6 \end{vmatrix}$

Expanding the above by the R_1 we get,

$\Delta = 4(12\text{-}21) - 1\,(54\text{-}42) + 5(27\text{-}12) = -36\text{-}12 + 75 = 27$

$\therefore\ \Delta = \Delta_1 = \Delta_2$

Hence, in all the above cases it is proved that by the interchange of any two adjacent rows or columns, the value of the determinant remains the same but with the opposite sign.

5. If every element in any row or column consists of the sum or difference of two quantities, then the determinant can be expressed a the sum or difference of two determinants of the same order:

(a) Let the determinant of order 2 be $|D| = \begin{vmatrix} 2+3 & 5 \\ 7+1 & 6 \end{vmatrix}$ and the value of the determinant b Δ.

$$\begin{aligned} \Delta &= |D|\ [(2 + 3) \times 6 - (7 + 1) \times 5] \\ &= (2 \times 6 + 3 \times 6 - 7 \times 5 - 1 \times 5) \\ &= (2 \times 6 - 7 \times 5) + (3 \times 6 - 1 \times 5) \\ &= \begin{vmatrix} 2 & 5 \\ 7 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} \end{aligned}$$

(b) Let the determinant of order 3 be $|D| = \begin{vmatrix} 1 + 2 & 1 & 5 \\ 3 + 6 & 3 & 6 \\ 2 + 4 & 2 & 7 \end{vmatrix}$ and the value of the determinant be Δ. $\therefore\ \Delta = |D| = (1+2)\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} - (3+6)\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + (2+4)\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}$

$$= 1\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} - 6\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 & 5 \\ 3 & 3 & 6 \\ 2 & 2 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 5 \\ 6 & 3 & 6 \\ 4 & 2 & 7 \end{vmatrix}$$

6. If each determinant in a row or a column of a determinant is multiplied by a constant K, then the value of the new determinant is K times the value of the original determinant.

Proof:

(a) Let the determinant of the order 2 be, $|A| = \begin{vmatrix} 4 & 5 \\ 9 & 7 \end{vmatrix}$ and the value of the determinant be Δ.

$$\Delta = \begin{vmatrix} 4 & 5 \\ 9 & 7 \end{vmatrix} = 4 \times 7 - 5 \times 9 = 28 - 45 = -17$$

Multiplying each element of Row – 1 by K = 3, we get,

$$K\ |A| = 3\begin{vmatrix} 4 & 5 \\ 9 & 7 \end{vmatrix} = \begin{vmatrix} 12 & 15 \\ 9 & 7 \end{vmatrix} = (84 - 135) = -51 = 3(-17) = 3\Delta$$

Notes

(b) Let determinant of the order 3 be, $\Delta = |D| = \begin{vmatrix} 1 & 4 & 5 \\ 3 & 6 & 6 \\ 2 & 9 & 7 \end{vmatrix}$ and the value of the determinant be Δ.

Expanding the above by the Row – 1 we get, Δ = |D| = 1 (42 – 54) – 4 (21 – 12) + 5 (27 – 12) = -12 – 36 + 75 = 27.

$$K|D| \begin{vmatrix} 1 \text{ x } 5 & 4 & 5 \\ 3 \text{ x } 5 & 6 & 6 \\ 2 \text{ x } 5 & 9 & 7 \end{vmatrix} = \begin{vmatrix} 5 & 4 & 5 \\ 15 & 6 & 6 \\ 10 & 9 & 7 \end{vmatrix}$$

Expanding the above determinant by the Row – 1 we get,

K |D| = 5 (42 – 54) – 4 (105 – 60) + 5 (135 – 60)

= 60 – 180 + 375 = 135 = 5 (27) = 5Δ

∴ K |D| = 5Δ

7. The value of the determinant remains unchanged, if to each element of any particular row or column of the determinant, the equimultiple of the corresponding elements of one or more rows or columns be added or subtracted.

Proof:

(a) Let the determinant of the order 2 be, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 8 \end{vmatrix}$ and the value of determinant be Δ.

Thus, Δ = |A| = (2 × 8 – 6 × 5) = -14

Adding 2 times the elements of Row – 2 to the corresponding elements of Row – 1 we get,

$$\Delta = |A| = \begin{vmatrix} 2 + 2 \times 6 & 5 + 2 \times 8 \\ 6 & 8 \end{vmatrix} = \begin{vmatrix} 14 & 21 \\ 6 & 8 \end{vmatrix} = (14 \times 8 - 6 \times 21) = 112 - 126 = -14$$

(b) Let the determinant of the order 3 be, $|D| = \begin{vmatrix} 4 & 16 & 5 \\ 3 & 12 & 7 \\ 2 & 8 & 8 \end{vmatrix}$ and the value of the determinant by Δ.

According to the property no. 2 stated above, the value of |D| = 0, since its C_1 and C_2 are identical. Now, adding 3 times the C_1 to the corresponding elements of C_2 we get,

$$\Delta = |D| = \begin{vmatrix} 4 & 4 + 3 \text{ x } 4 & 5 \\ 3 & 3 + 3 \text{ x } 3 & 7 \\ 2 & 2 + 3 \text{ x } 2 & 8 \end{vmatrix} = \begin{vmatrix} 4 & 16 & 5 \\ 3 & 12 & 7 \\ 2 & 8 & 8 \end{vmatrix}$$

Expanding the above determinant by the R_1 we get,

Δ = |D| = 4 (12 × 8 – 8 × 7) – 16 (3 × 8 – 2 × 7) + 5 (3 × 8 – 2 × 12)

= 4 (96 – 56) – 16 (24 – 14) + 5 (24 – 24)

= 160 – 160 + 0 = 0

(c) Let the determinant of the order 3 be,

$|D| = \begin{vmatrix} 16 & 4 & 5 \\ 12 & 3 & 7 \\ 8 & 2 & 8 \end{vmatrix}$ and the value of the determinant be Δ.

Notes

Expanding the above along the R_1 we get,

$\Delta = |D| = 16\ (3 \times 8 - 2 \times 7) - 4\ (12 \times 8 - 8 \times 7) + 5\ (12 \times 2 - 8 \times 3)$

$= 16\ (24 - 14) - 4\ (96 - 56) + 5\ (24 - 24)$

$= 160 - 160 + 0 = 0$

Subtracting 3 times the elements of the C_2 from the corresponding elements of the C_1 we get,

$$\Delta = |D| = \begin{vmatrix} 16 - 3 \text{ x } 4 & 4 & 5 \\ 12 - 3 \text{ x } 3 & 3 & 7 \\ 8 - 3 \text{ x } 2 & 2 & 8 \end{vmatrix} = \begin{vmatrix} 4 & 4 & 5 \\ 3 & 3 & 7 \\ 2 & 2 & 8 \end{vmatrix} = 0 \quad [\therefore C_1 \text{ \& } C_3 \text{ are identical}]$$

Hence in all the above examples, the property of the determinants cited above hold good.

8. **If the elements of any row or column of a determinant are multiplied in order by the cofactor C_{ij} of the corresponding elements of any other row or column, then the sum of the products thus obtained is zero.**

Proof:

(a) Let the determinant of the order – 3 be, $|A| = \begin{vmatrix} 4 & 7 \\ 3 & 8 \end{vmatrix}$

The cofactors of the elements of the second row are:

$C_{21} = (-1)^2 + 1 \times 7 = -1 \times 7 = -7$

$C_{22} = (-1)^2 + 2 \times 4 = 1 \times 4 = 4$

Now, multiplying the elements of the R1 by the cofactors of their corresponding elements of the second row and getting them summed up we get, $4 \times -7 + 7 \times 4 = -28 + 28 = 0$

(b) Let the determinant of the order 3 × 3 be, $|D| = \begin{vmatrix} 4 & 7 & 3 \\ 8 & 0 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

The cofactors of the elements of the C_2 are:

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 8 & 5 \\ 2 & 1 \end{vmatrix} = -1(8 - 10) = 2$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 1(4 - 6) = -2$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 4 & 3 \\ 8 & 5 \end{vmatrix} = -1(20 - 24) = 4$$

Now, multiplying the elements of the C_3 by the cofactors of the corresponding elements of the C_2 and getting them summed up we get, $3 \times 2 + 5 \times -2 + 1 \times 4 = 6 - 10 + 4 = 0$

Again, multiplying the elements of the C_1 by the cofactors of the corresponding elements of the C_2 and getting them totalled we get, $4 \times 2 + 8 \times -2 + 2 \times 4 = 8 - 16 + 8 = 0$

Hence, from the results of all the above examples the property of the determinant thus cited is proved.

Notes

Illustrations on Uses of the Properties

The following illustrations will show how determinants are found out easily by using their appropriate properties:

Illustration 1: Using the appropriate properties evaluate the determinants

(i) $\begin{vmatrix} 1 & 3 & 4 \\ 5 & 15 & 16 \\ 6 & 18 & 8 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 6 & 12 \\ 6 & 9 & 21 \\ 9 & 12 & 30 \end{vmatrix}$ (iii) $\begin{vmatrix} 1+\omega & \omega^2 \\ 1+\omega^2 & \omega \end{vmatrix}$ (Given, $1 + \omega + \omega^2 = 0$)

Solution:

(i) $$\Delta = \begin{vmatrix} 1 & 3 & 4 \\ 5 & 15 & 16 \\ 6 & 18 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 \text{ x } 1 & 4 \\ 5 & 3 \text{ x } 5 & 16 \\ 6 & 3 \text{ x } 6 & 8 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & 4 \\ 5 & 5 & 16 \\ 6 & 6 & 18 \end{vmatrix}$$

= 3 × 0 = 0 (refer property 6 and 2)

(ii) $$\Delta = \begin{vmatrix} 3 & 6 & 12 \\ 6 & 9 & 21 \\ 9 & 12 & 30 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 3 \\ 6 & 9 & 3 \\ 9 & 12 & 3 \end{vmatrix} C_3 \to C_3 - 3C_1 = \begin{vmatrix} 3 & 3 & 3 \\ 6 & 3 & 3 \\ 9 & 3 & 3 \end{vmatrix} C_2 \to C_2 - C_1$$

= 0 (refer property 2)

(iii) $$\Delta = \begin{vmatrix} 1+\omega & \omega^2 \\ 1+\omega^2 & \omega \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & \omega^2 \\ 1+\omega^2+\omega & \omega \end{vmatrix} C_1 \to C_1 + C_2 = \begin{vmatrix} 0 & \omega^2 \\ 0 & \omega \end{vmatrix} = 0 \text{ (Property } - 1)$$

Remark: In complex Number ω is considered as cube root of unity. Thus $\omega = \sqrt[3]{1}$ or $\omega^3 = 1$ and $\omega^3 - 1 = 0$, So, $(\omega - 1)(\omega^2 + \omega + 1) = 0$, $\therefore (\omega^2 + \omega + 1) = 0$

1.10 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS USING CRAMER'S RULE

In linear algebra, Cramer's rule is an explicit formula for the solution of a system of linear equations and is valid whenever the system has a unique solution. French mathematician Gabriel Cramer (1704-1752) discussed the method in his publication on the rule for the arbitrary number of unknown's in the year 1750.

System of equations of two unknown values

Theorem 1: (Cramer's Rule)

The solution of the system of equations

$a_1x + b_1y = c_1$ …(i)

$a_2x + b_2y = c_2$ …(ii)

is given by $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$,

where, $$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_y \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ and } D \neq 0$$

Notes

Remember:

Here, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ = determinant of the coefficients of x and y

To obtain D_x , replace a_1 , a_2 by c_1, c_2 respectively in D.

To obtain D_y , replace b_1 , b_2 by C_1 , C_2 respectively in D.

Illustration 2: Solve the following system of equations through determinants using Cramer's Rule:

(i) 4x + 3y = 8
6x + 7y = 17

(ii) 3x + 4y = 5
x – y = -3

Solution:

(i) The given equations are:

4x + 3y = 8

6x + 7y = 17

We have, $D = \begin{vmatrix} 4 & 3 \\ 6 & 7 \end{vmatrix} = 4 \text{ x } 7 - 6 \text{ x } 3 = 28 - 18 = 10$

$D_x = \begin{vmatrix} 8 & 3 \\ 17 & 7 \end{vmatrix} = 8 \text{ x } 7 - 17 \text{ x } 3 = 56 - 51 = 5$

And $D_y = \begin{vmatrix} 4 & 8 \\ 6 & 17 \end{vmatrix} = 4 \text{ x } 17 - 6 \text{ x } 8 = 68 - 48 = 20$

Applying the Cramer's rule we get,

$$x = \frac{D_x}{D} = \frac{5}{10} = \frac{1}{2} \text{ or } 0.5$$

$$y = \frac{D_y}{D} = \frac{20}{10} = 2$$

Hence, x = 0.5 and y = 2

(ii) The given equations are:

3x + 4y = 5

x – y = -3

We have, $D = \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -(3 + 4) = -7$

$D_x = \begin{vmatrix} 5 & 4 \\ -3 & -1 \end{vmatrix} = -(5 - 12) = 7$

And $D_y = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -(9 + 5) = -14$

Applying the Cramer's rule we get,

$$x = \frac{D_x}{D} = \frac{7}{-7} = -1$$

Notes

And $y = \frac{D_y}{D} = \frac{-14}{-7} = 2$

Hence x = –1 and y = 2

System of equations of three unknown values

THEOREM 2. (Cramer's Rule)

The solution of the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

is given by $x = \frac{D_x}{D}, y = \frac{Dy}{D}$ and $z = \frac{D_z}{D}$

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}; D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ and $D \neq 0$

Remember:

Here, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

To obtain D_x replace a_1 , a_2 , a_3 by d_1 , d_2 , d_3 respectively in D.

To obtain D_y replace b_1 , b_2 , b_3 by d_1 , d_2 , d_3 respectively in D.

To obtain D_z replace C_1 , C_2 , C_3 by d_1 , d_2 , d_3 respectively in D.

Illustration 3: Solve the following linear equations with the help of Cramer's rule:

(i) 2x + 3y = 3 (ii) x + 2y + 3z = 6

3x + 2y = 7 2x + y + z = 4

X + y + 2z = 4

Solution:

(i) We have,

$D = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 2 \text{ x } 2 - 3 \text{ x } 3 = 4 - 9 = -5$

$D_x = \begin{vmatrix} 3 & 3 \\ 7 & 2 \end{vmatrix} = 3 \text{ x } 2 - 3 \text{ x } 7 = 6 - 21 = -15$

$D_y = \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 2 \text{ x } 7 - 3 \text{ x } 3 = 14 - 9 = 5$

Notes

Hence, $x = \frac{D_x}{D} = \frac{-15}{-5} = 3$ and $y = \frac{D_y}{D} = \frac{5}{-5} = -1$

(ii) We have,

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1\ (1 \times 2 - 1 \times 1) - 2\ (2 \times 2 - 1 \times 1) + 3\ (2 \times 1 - 1 \times 1)$$

$$= 1 \times 1 - 2 \times 3 + 3 \times 1 = -2$$

$$D_x = \begin{vmatrix} 6 & 2 & 3 \\ 4 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 6\ (1 \times 2 - 1 \times 1) - 2\ (4 \times 2 - 1 \times 4) + 3\ (4 \times 1 - 1 \times 4)$$

$$= 6 \times 1 - 2 \times 4 - 3 \times 0 = -2$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 4 & 1 \\ 1 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 4 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 1 \times 4 - 6 \times 3 + 3 \times 4 = -2$$

Hence, $x = \frac{D_x}{D} = \frac{-2}{-2} = 1 \quad Y = \frac{D_y}{D} = \frac{-2}{-2} = 1$

Notes

MATRICES

1.11 MEANING OF MATRICES

A set of conditions that provides a system in which something grows or develops is called matrix. Mathematically, a group of numbers or other symbols arranged in a rectangle that can be used together as a single unit to solve particular mathematical problems is called matrix. The mathematical use of the term matrix was first introduced in 1850 by James Joseph Sylvester. But the contribution of Carl Fridrich Gauss (1771 – 1855) and Gottfired Leibniz (1646 – 1716) in the development of Matrix Algebra cannot be ignored. Now, matrices have become an useful tool in solving business problems.

1.12 DEFINITION OF MATRICES

A matrix may be defined as an orderly arrangement of some number of symbols in certain rows and columns enclosed by some brackets, subscripted by the magnitude of its order and denominated by some capital letter. In other words, a matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets and subject to certain rules of presentation.

The following are the specimens of matrix:

$$\text{(i)} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}_{3x3} \quad \text{(ii) } B = \begin{vmatrix} 15 & 18 \\ 20 & 15 \\ 30 & 40 \end{vmatrix}_{3x2} \quad \text{(iii) } C = \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix}_{2x3}$$

1.13 ESSENTIAL CHARACTERISTICS OF MATRICES

From the above definition and the specimens, the essential characteristics of a matrix may be analysed as under:

(i) **It consists of some numbers of symbols.** The numbers like 0,5, 10, 125, 3500 and the symbols like x, y, z etc. constitute a matrix, These are called the elements of a matrix without which a matrix cannot come into existence. These numbers may take any sign and any form like, 0.35, 0.75, ± fractions like $\frac{3}{7}, \frac{-2}{11}, \frac{-7}{9}$ and ± mixed numbers like 10.75, - 3.375 etc. They may consist of single digits or multiple digits including only zeroes even. However in order to constitute a matrix, they must be orderly arranged in some rows and columns. Any disorderly scattered numbers or symbols will not constitute a matrix. For example, the following groups of numbers and symbols will not amount to matrices:

$$\text{(a)} \begin{bmatrix} & 2 & & 3 \\ 1 & 6 & 5 & 4 \\ & 8 & 7 & 9 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} & P & \\ y & x & z \\ & n & m \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 5 & \\ 3 & 3 & 4 \\ 4 & & 7 \end{bmatrix}$$

Further, the elements should be so arranged that each of them is capable of being subscripted by its ith row and jth column to locate its position in the matrix. Thus, if an element say, 15 subscripted as 15_{12} it would indicate that the said element lies in the first row and second column of the matrix. In a disorderly arrangement of numbers like the above ones a number cannot be subscripted by its ith two and jth column.

(ii) **It consists of some rows and columns:** A matrix always consists of certain rows and columns in which all its elements are arranged. The number of such rows and columns may be one or more and there may or may not be equality between the number of rows and the number of columns. But a column or a row must be complete with some elements.

Notes

Thus a group of rows and columns not completed with all its elements as follows will not amount to a matrix:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix}$ (a) $\begin{bmatrix} 1 & 5 \\ 2 & \\ 3 & 4 \end{bmatrix}$

It may be noted that an empty space in a row or a column is not equal to '0' for that a '0' is an element, whereas an empty space is never an element of a matrix.

(iii) It must be enclosed by some brackets. A group of orderly arranged numbers or symbols to be called a matrix must be enclosed by some brackets viz. parentheses ( ), square brackets [ ], or curly brackets { }, However, conventionally the curly brackets are not used in representing a matrix.

Thus, a group of following numbers and symbols not encompassed by any bracket will not constitute a matrix:

(a) $\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}$ (b) $\begin{matrix} x & y & z \\ m & n & p \end{matrix}$ (c) $\begin{matrix} 4 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{matrix}$ (d) $\begin{matrix} 3 & 2 \\ 5 & 4 \end{matrix}$

(iv) It must be subscripted by the magnitude of its order. The magnitude of the order of a matrix refers to the number of rows and columns with which a matrix is constituted. The number of rows and columns may be subscripted at the bottom of the right hand side bracket of a matrix as m × n (read as m by n), where m, represents the number of rows and on the number of columns in the matrix. Thus, in a matrix, if the subscript stands like, 4 × 3, it will mean that there are 4 rows and 3 columns in the said matrix.

(v) It must be denominated by some capital letter. Every matrix must be denominated properly for making a reference to it in the course of computational works. Conventionally, all the matrices are denominated or named by some letters of upper case viz. A, B, C, D etc. Without the proper denomination, any orderly arrangement of numbers or symbols will not constitute a matrix.

Having thus analysed, the whole corpus of a matrix may be represented as under:

$$A \begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ a_{ij} & a_{ij} & a_{ij} \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix}_{m \times n}$$

Where, 'A' refers to the name of the matrix, 'a' to the element or entry in the matrix, i.j. to the subscript of an element in which i, indicates the row and j the column of the matrix in which the element appears.

Order of a Matrix

m × n, refers to the subscript of the matrix in which m indicates the number of rows and n the number of columns contained in the matrix and ( ), to the enclosure or boundary of the matrix.

Besides, the horizontal lines and the vertical lines in which the elements stand orderly placed are respected called rows and columns of the matrix.

1.14 DIFFERENT TYPES OF MATRIX

Before entering upon the operation on matrices, it is highly necessary to have an idea about the various types of forms of matrix.

These are identified here as under:

(i) **Row Matrix.** A Matrix that appears with one row only is called a row matrix.

Examples : (i) $A = [0\ 1\ 2]_{1\times3}$; $B = [12\ 22\ 15]_{1\times4}$

Notes

Examples : (i) $\begin{bmatrix}0\\1\\0\end{bmatrix}_{3\times1}$ (ii) $B = \begin{bmatrix}0\\1\\2\\3\end{bmatrix}_{4\times1}$ (iii) $C = \begin{bmatrix}15\\25\\35\\45\\40\end{bmatrix}_{5\times1}$

(iii) **Null (or Zero Matrix.** A matrix that consists of zeroes only is called a Null or Zero matrix. This us usually denoted by the capital letter O and it is also popularly known as Null matrix.

Examples: (i) $O = \begin{bmatrix}0\\0\end{bmatrix}_{2\times1}$ (ii) $O = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}_{2\times3}$ (iii) $\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}_{3\times4}$

(iv) **Singleton matrix.** A matrix that comprises one element only is called a singleton matrix.

Examples. (i) $A = (0)_{1x1}$ (ii) $B = (5)_{1x1}$ (iii) $C = (25)_{1x1}$ (iv) $D = (105)_{1x1}$

Here, it may be noted that 0, is an element; 25 is a single number, though it consists of two digits; 105 is also a single number, though it is made of three digits.

(v) **Square matrix.** A matrix that appears with equal number of rows and columns (i.e.m = n) is called a square matrix.

Examples: (i) $A = (0)_{1\times1}$, (ii) $B = \begin{bmatrix}1&2\\3&4\end{bmatrix}_{2\times2}$ (iii) $C = \begin{bmatrix}10&11&12\\13&14&15\\16&17&18\end{bmatrix}_{3x3}$

(vi) **Diagonal Matrix.** A square matrix in which all the principal diagonal elements are non-zeroes and all other elements are zeroes is called a diagonal matrix.

Examples:

(i) $A = 5\begin{bmatrix}1&0\\0&2\end{bmatrix}_{2\times2}$ (ii) $B = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}_{3x3}$ (iii) $C = \begin{bmatrix}9&0&9&0\\0&8&0&0\\0&0&7&0\\0&0&0&6\end{bmatrix}_{4x4}$

(iv) $D = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{33}&a_{33}\end{bmatrix}_{4x4}$

Note: Principal diagonal element. An element, both the subscripts (i and j) of which are equal is called a principal diagonal element. The line along which the principal diagonal elements are positioned is called the principal or leading diagonal.

Exmaple: a_{11}, a_{22}, a_{33}, a_{44} and the like.

The sum of the principal diagonal elements of a square matrix is called Trace. In example (ii), the trace is 1 + 2 + 3 = 6 similarly, in matrix, $A = \begin{bmatrix}2&3\\4&5\end{bmatrix}$, the trace is 2 + 5 = 7.

(vii) **Scalar Matrix.** A diagonal matrix in which all the leading diagonal elements are equal is called a Scalar Matrix. In other words, a square matrix in which all the elements except those in the main diagonal are zeros and all the leading diagonal elements are equal is called a scalar matrix.

Examples: (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2}$ (ii) B = $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times 3}$

(viii) **Identity (or Unity) matrix.** A square matrix in which all the leading diagonal elements are unity or 1 and all other elements are zeroes is called a identity or unity matrix. It is conventionally denoted by the capital letter, I.

Examples: (I) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ (ii) I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ (iii) I = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$

(ix) **Triangular matrix.** A square matrix in which all the elements above or below the principal diagonal are zeroes, and the rest are non-zeroes is called a triangular matrix. If the zero elements lie below the principal diagonal, it is called an upper-triangular matrix, and if the zero elements lie above the principal diagonal it is called a lower triangular matrix.

Examples: (i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times 3}$ (ii) 1 = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 7 & 8 & 3 & 0 \\ 9 & 4 & 6 & 4 \end{bmatrix}_{4\times 4}$

An upper triangular marix A lower triangular matrix

(x) **Equal matrix.** A matrix is said to be equal to another matrix, if all its elements are equal to the corresponding elements of the said another matrix.

Examples: (i) If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ and B = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 2}$, then A = B

$\therefore$ A is an equal matrix to B and vice versa.

(ii) If C = $\begin{bmatrix} 1 & 7 \\ 4 & 3 \end{bmatrix}_{2\times 2}$ and D = $\begin{bmatrix} x & x + y \\ 4 & 3 \end{bmatrix}_{2\times 2}$

When, x = 1, and y = 6, then C = D

$\therefore$ C is an equal matrix to D and vice versa

Example 1: If $\begin{bmatrix} a & 4 \\ 2a + 3b & 7 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 20 & 7 \end{bmatrix}$ find a and b.

Solution:

Given $\begin{bmatrix} a & 4 \\ 2a + 3b & 7 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 20 & 7 \end{bmatrix}$

$\therefore$ By the equality of the corresponding elements we get,

a = 7 and 2a + 3b = 20

$\Rightarrow 2 \times 7 + 3b = 20 \Rightarrow b = (20 - 14)/3 = 2$

Hence, a = 7 and b = 2

Notes

(xi) **Comparable matrices or equivalent matrices.** A matrix is said to be comparable or equivalent to another matrix, if the number of its rows and columns is equal to those of the other matrix i.e. $m_1 = m_2$ and $n_1 = n_2$

Examples:

If $A = \begin{bmatrix} 1 & 7 & 8 \\ 3 & 1 & 2 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 15 & 10 & 25 \\ 16 & 8 & 14 \end{bmatrix}_{2\times3}$ then A ~ B

(xii) **Sub matrix.** A small matrix obtained by deleting some rows or (and) some columns of a given matrix is called a sub-matrix.

Examples:

If the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ following will be the examples of its submatrices.

$A_1 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_{2\times2}$ $B_1 = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}_{2\times2}$ $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$ $A_4 = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}_{2\times2}$

1.15 ARITHMETIC OPERATIONS ON MATRICES

The basic arithmetic operations of addition, subtraction, multiplication and division can very well be performed on matrices subject to certain conditions and procedures laid down as under:

(i) Addition of Matrices

Condition necessary

The matrices to be added to each other must be comparable i.e. each of the matrices must have equal number of rows and columns. Symbolically, $m_1 = m_2 = m_3$ and so on, and $n_1 = n_2 = n_3$ and so on.

Procedure

Place all the matrices to be added in a horizontal line and put + signs between each of the pairs of them.

Add the corresponding elements of each of the matrices and put their sums in the same order.

Example 2: Find the sum of addition of the following matries:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}_{3\times3}$

Solution:

The condition of addition is satisfied as each of the two given matrices is in the same order. Thus, placing the two matrices in a horizontal line we get,

(i) $A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$

Adding the corresponding elements in each of the matrices we get,

Notes

$$\begin{bmatrix} 1+10 & 2+11 & 3+12 \\ 4+13 & 5+14 & 6+15 \\ 7+16 & 8+17 & 9+18 \end{bmatrix} = \begin{bmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{bmatrix}_{3x3}$$

(ii) $A + B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}_{2x2}$

Example 3: Find the sum of addition of the following three matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}_{2x4} \quad B = \begin{bmatrix} 8 & 9 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}_{2x4} \quad \text{and } C = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 7 & 7 & 9 \end{bmatrix}_{2x4}$$

Solution:

The condition of matrix addition is satisfied as each of the given matrices is of the same order 2 × 4. Thus, placing the three matrices in a horizontal line we get.

$$A + B + C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 8 & 9 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 7 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8+4 & 2+9+3 & 3+7+2 & 4+6+1 \\ 5+5+5 & 6+4+7 & 7+3+6 & 8+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 14 & 12 & 11 \\ 15 & 17 & 16 & 19 \end{bmatrix}_{2x4}$$

Properties of Matrix Addition

It may be noted that matrix addition has the following important properties of which one may take advantage in the matter of computations:

(a) It is commutative. This means, A + B = B + A

(b) It is associative. This means (A+B) + C = A + (B+C)

(c) It has additive identity. This means, A + O = O + A = A

(d) It has additive inverse. This means A + -A = -A + A = 0

(ii) Subtraction of Matrices

Condition Necessary

Both the matrices i.e. the subtrahenal and the minuend matrices must be equivalent to each other. This means that each of the matrices must have equality in respect of the number of their rows and columns.

Procedure

Place both the matrices in a horizontal line and put a – ve sign between the minuend and the subtrahenal matrices.

Subtract the elements of the subtrahenal matrix from their corresponding elements in the minuend matrix and put their sums in the same order.

Example 4: Subtract the matrix, B from the matrix, A where,

$$A = \begin{bmatrix} 6 & 8 \\ 2 & -3 \end{bmatrix}_{2x2} \quad B = \begin{bmatrix} 2 & 6 \\ 5 & -2 \end{bmatrix}_{2x2}$$

Notes

$$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 7 & 2 & 3 \end{bmatrix}_{3x3} \quad B = \begin{bmatrix} 5 & 4 & 3 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{3x3}$$

Solution:

The condition of matrix subtraction is satisfied as both the matrices given, are of the same order. Now placing both the matrices in a horizontal line as follows, we get

(i) $$A - B = \begin{bmatrix} 6 & 8 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 6-2 & 8-6 \\ 2-5 & -3-(-2) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}_{2x2}$$

(ii) $$A - B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 4 & 3 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{3x3} = \begin{bmatrix} 9-5 & 8-4 & 7-3 \\ 6-1 & 5-2 & 4-3 \\ 1-5 & 2-6 & 3-7 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 5 & 3 & 1 \\ -4 & -4 & -4 \end{bmatrix}_{3x3}$$

Example 5: Find the sum of the following operations on the matrices, A + B – C, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 9 \end{bmatrix}_{3x3} \quad B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{3x3} \quad \text{and } C = \begin{bmatrix} 4 & 6 & 3 \\ 3 & 5 & 4 \\ 3 & 0 & 1 \end{bmatrix}_{3x3}$$

Solution:

The conditions of both addition and subtraction of matrices are satisfied as all of the matrices given are of the same order 3 × 3.

Placing the matrices given, in a horizontal line we get,

$$A + B - C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 3 \\ 3 & 5 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2-4 & 1+5-6 & 1+3-3 \\ 1+3-3 & 2+1-5 & 3+2-4 \\ 1+1-3 & 4+2-0 & 9+1-1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 6 & 9 \end{bmatrix}_{3x3}$$

(iii) Multiplication of Matrices

There can be two types of multiplication with the matrices. They are :

(a) Scalar multiplication and (b) multiplication proper.

These are explained here as under:

(a) Scalar multiplication:

When each element of a matrix is multiplied by a constant called a scalar, it is called scalar multiplication.

Condition necessary

No condition as to the order of a matrix is necessary except that the scalar quantity must have been given.

Procedure

(i) Write the scalar first, and then place the given matrix adjacent to it without putting any algebraic sign between them.

Notes

(ii) Multiply each element of the given matrix by the scalar given, and put the respective products in the same order.

Example 5: Find the product of the scalar multiplication with the following:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3x4} \quad \text{and } K = 5.$$

Solution:

Placing the scalar K and the matrix A in the multiplication form, we get,

$$KA = 5\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Multiplying each element of the matrix by the scalar 5 we get,

$$KA = \begin{bmatrix} 5x1 & 5x2 & 5x3 & 5x4 \\ 5x5 & 5x6 & 5x7 & 5x8 \\ 5x9 & 5x10 & 5x11 & 5x12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 15 & 20 \\ 25 & 30 & 35 & 40 \\ 45 & 50 & 55 & 60 \end{bmatrix}_{3x4}$$

Properties of Scalar Multiplication

(i) It is distributive over addition

This implies that K (A + B) = KA + KB

Example 6: From the following data prove that the scalar multiplication of matrices is distributive over addition.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{bmatrix}_{2x3}, \quad R = \begin{bmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{bmatrix}_{2x3}, \quad \text{and } K = 8,$$

Solution:

According to the property of scalar multiplication, we have

K (A + B) = KA + KB

Where

$$K (A + B) = 8\left[\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{bmatrix}\right]$$

$$= 8\begin{bmatrix} 1+4 & 2+5 & 3+7 \\ 5+2 & 7+9 & 8+8 \end{bmatrix} = 8\begin{bmatrix} 5 & 7 & 10 \\ 7 & 16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \text{ x } 5 & 8 \text{ x } 7 & 8 \text{ x } 18 \\ 8 \text{ x } 7 & 8 \text{ x } 18 & 8 \text{ x } 16 \end{bmatrix} = \begin{bmatrix} 40 & 56 & 80 \\ 56 & 125 & 128 \end{bmatrix}_{2x3}$$

And KA + KB

$$= 8\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{bmatrix} + 8\begin{bmatrix} 4 & 5 & 7 \\ 2 & 9 & 8 \end{bmatrix}$$

Notes

$$= \begin{bmatrix} 8 \text{ x } 1 & 8 \text{ x } 2 & 8 \text{ x } 3 \\ 8 \text{ x } 5 & 8 \text{ x } 7 & 8 \text{ x } 8 \end{bmatrix} + \begin{bmatrix} 8 \text{ x } 4 & 8 \text{ x } 5 & 8 \text{ x } 7 \\ 8 \text{ x } 2 & 8 \text{ x } 9 & 8 \text{ x } 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 16 & 24 \\ 40 & 56 & 64 \end{bmatrix} + \begin{bmatrix} 32 & 40 & 56 \\ 16 & 72 & 64 \end{bmatrix}$$

$$= \begin{bmatrix} 8+32 & 16+40 & 24+56 \\ 40+16 & 56+72 & 64+64 \end{bmatrix} = \begin{bmatrix} 40 & 56 & 80 \\ 56 & 128 & 128 \end{bmatrix}_{2\text{x}3}$$

Thus, it is proved that K(A+B) = KA + KB

(b) Multiplication Proper (or Multiplication of Matrices)

The multiplication among two matrices is possible only when the number of column of the 1st matrix is equal to the number of rows of the 2nd matrix. In other words, a matrix A is conformable to another matrix B for multiplication i.e. AB exists, only when the number of columns in A equals to the number of rows in B.

Procedure

(i) Place the matrices in a horizontal line without putting any sign between them.

(ii) Multiply each element of the first row of the multiplicand by the corresponding element of the first column of the multiplier, and get them totalled to obtain the first element of the first row of the product matrix.

(iii) Similarly, multiply each element of the first row of the multiplicand by the corresponding element of the nth column of the multiplier and get them totalled to obtain the nth element of the first row of the product matrix.

Continue the above procedure to obtain the elements of the other rows of the product matrix.

Example 6: Find the product of the two matrices A and B where,

(i) $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\text{x}2}$ $B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}_{2\text{x}1}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\text{x}2}$ $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}_{2\text{x}2}$

Solution:

The condition of multiplication proper is satisfied by the given matrices, in both the cases since the number of columns in the multiplicand matrix A (prefactor) is equal to the number of rows in the multiplier matrix B (post factor), in both the cases.

Placing the matrices in a horizontal line we get,

(i) $AB = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \text{ x } 4 + 3 \text{ x } -1 \\ 2 \text{ x } 4 + 1 \text{ x} -1 \end{bmatrix} - \begin{bmatrix} 4-3 \\ 8-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}_{2\text{x}1}$$

(ii) $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ x } \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 \text{ x } 2 + 2 \text{ x } 5 & 1 \text{ x } 3 + 2 \text{ x } 1 \\ 3 \text{ x } 2 + 4 \text{ x } 5 & 3 \text{ x } 3 + 4 \text{ x } 1 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 26 & 13 \end{bmatrix}$

Properties of Multiplication Proper

It may be noted that the matrix multiplication has the following important properties of which one may take advantage in the course of computational works.

(a) It is not commutative. This means AB ≠ BA (always)

(b) It is associative. This means (AB)C = A (BC)

Notes

(c) It is distributive over addition. This means A (B+C) = AB + AC

(iv) Division of Matrices

Condition necessary

The number of columns in the dividend matrix (n_1) must be equal to the number of rows in the divisor matrix (m_2).

Proceed with the work of division just on the lines of multiplication proper explained above, except that each element in the dividend is to be multiplied by the reciprocal of the corresponding element of the divisor matrix (i.e. $\frac{1}{a}$)

Illustration 7: Divide the matrix A by the matrix B where,

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2x3} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3x2}$$

Solution:

Proceeding with the procedure of multiplication proper we get,

$$A{:}B = \begin{bmatrix} 4 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} : \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \text{ x } 1/1 + 2 \text{ x } 1/3 + 3 \text{ x } 1/5 & 4 \text{ x } 1/2 + 2 \text{ x } 1/4 + 3 \text{ x } 1/6 \\ 4 \text{ x} 1/1 + 5 \text{ x } 1/3 + 6 \text{ x } 1/5 & 4 \text{ x } 1/2 + 5 \text{ x } 1/4 + 6 \text{ x } 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 2/3 + 3/5 & 2 + 1/2 + 1/2 \\ 4 + 5/3 + 6/5 & 2 + 5/4 + 1 \end{bmatrix} = \begin{bmatrix} 79/15 & 3 \\ 103/15 & 17/4 \end{bmatrix}_{2x2}$$

1.16 TRANSPOSE OF MATRICES

A matrix which is obtained by changing the rows into their respective columns or the columns into their respective rows of a matrix is called a transposed matrix. To obtain such matrix, the first row (R_1) of a given matrix is made the first column (C_1), the second row (R_2) is made the second column (C_2) and so on, or the first column (C_1) is the first row (R_1), the second column (C_2) is made the second row (R_2) and so on.

The transposed matrix is denoted by A' or A^t etc.

Examples: (i) If A = $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 3 \end{bmatrix}$; then $A' = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 3 \end{bmatrix}$

If B = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; then $B' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Note. Transpose of a transposed matrix reproduces the original matrix.

Notes

1.17 SOME SPECIAL FORM OF SQUARE MATRICES

(i) ***Orthogonal Matrix:*** A square matrix, which when multiplied by its transpose amounts to an identity matrix is called an orthogonal matrix. Thus, if $A \times A' = 1$ then A is orthogonal matrix.

(ii) ***Symmetric Matrix:*** A square matrix A is said to be symmetric if $A' = A$

Examples: $\begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are symmetric matrix

(iii) ***Skew-Symmetric Matrix:*** A square matrix A is said to be Skew – Symmetric matrix, if A' = -A.

Note: The matrix, $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ is Skew − Symmetric

Since $A' = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$

(iv) ***Indempotent Matrix:*** A symmetric matrix that reproduces itself is termed as an indempotent matrix, if AA = A

Note. The marix, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a symmetric marix. Now $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus AA = A

1.18 INVERSE OF A MATRIX

Singular and Non-Singular Matrices

A square marix A is said to be singular if $|A| = 0$ and it is said to be non-singular if $|A| \neq 0$

Example:

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 0$

If $B = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} = -2 \neq 0$

Thus, matrix is a Non-singular matrix.

Invertible Matrix

A non zero square matrix A of order n is said to be invertible if there exists a square matrix B of order n, such that $AB = BA = I_n$.

We can say that the inverse of A is B, or $A^{-1} = B$

When, $AB = BA = I_n$, we have $A^{-1} = B$ and $B^{-1} = A$

Theorem 2.

A square matrix A is invertible if and only if A is non-singular i.e. A is invertible $\Leftrightarrow |A| \# 0$

Notes

Proof:

Let A be non-singular, then $|A| \neq 0$

$\therefore$ According to Theorem -1, A. (Adj A) = (Adj A). A = $|A|. I_n$

$$\Rightarrow \quad A\left(\frac{1}{|A|}.AdjA\right) = \left(\frac{1}{|A|}.AdjA\right)A = I_n \text{ (Multiplying } \frac{1}{|A|} \text{ both the sides)}$$

$$\Rightarrow \quad A\left(\frac{1}{|A|}.AdjA\right) = I_n \text{ or } \left(\frac{1}{|A|}.AdjA\right)A = I_n$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}.AdjA$$

Thus, it shows that whenever matrix A is non-singular, then matrix A is invertible

Finding Inverse of a Matrix (A^{-1})

The concept of inverse matrix is useful in solving the business problems expressed in simultaneous equations. Inverse of a square matrix can be find out by applying various methods, but

(i) Adjoint method (or co-factor method) and

(ii) Elementary operation method (or e-operation method) are very much popular among others.

(i) ***Using Adjoint Matrices (Co-factor Method):*** From the above discussion, it is clear that the inverse of a matrix can be obtained only if it is a non-singular square matrix and the formula to find out the inverse of A matrix is

$$A^{-1} = \frac{1}{|A|}.AdjA$$

Procedure

The following steps are to be taken up in turn of determining the inverse of a matrix.

Evaluate the determinant of the matrix. If it is zero, then stop proceeding further, for in that case, inverse does not exist for the matrix.

Find the adjoint of the matrix by the procedure detailed earlier.

Divide the adjoint of the matrix by its determinant and get the quotient as the inverse of the said matrix. Symbolically, the inverse of a matrix is given $A^{-1} = \frac{1}{|A|}.AdjA$

i.e. Inverse of the matrix $A = \frac{1}{\text{Determine of the matrix A}}$ x Adjoint of the matrix A

1.19 SOLUTION OF SIMULTANEOUS LINEAR EQUATION BY MATRIX ALGEBRA

The following steps need to be taken up in turn for solving the simultaneous equations involving upto 3 variables.

Steps

(i) Present the given equations in three different matrices, viz (a) matrix of the coefficients of the variables denoted by A, (b) matrix of the variables viz. x, y, z denoted by X, and (c) matrix of the constants denoted by B in the same order in which they stand in the equations i.e. AX = B.

Notes

(ii) Multiply A^{-1} on both the sides of the matrix equation and reduce the same to the following solution equation.

$X = A^{-1} . B$ $\quad [\therefore A^{-1} . AX = A^{-1} . B]$

(iii) Find the A^{-1} in accordance with its procedure detailed earlier.

(iv) Multiply the matrix A^{-1} with the matrix B and present the matrix of the product $A^{-1} . B$ thus obtained vis-à-vis the matrix of the variable X. These matrices must be equal to each other for which their corresponding elements would be equal to each other.

(v) Locate the value of each variable in the matrix X with reference to the corresponding element in the product matrix. The following illustrations will show how simultaneous equations are solved using the technique of matrix algebra.

Illustration 8: Using the technique of matrix algebra solve the following simultaneous equations:

$4x + 3y = 8$

$6x + 7y = 17$

Solution:

Presenting the given equations orderly in three different matrices as under we have,

$$\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$$

i.e. $A . X = B$

Where, $A = \begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$

By the technique of inverse we have, $X = A^{-1}B$

Where, $A^{-1} = \frac{1}{|A|}$ Adj. and $|A| = \begin{vmatrix} 4 & 3 \\ 6 & 7 \end{vmatrix} = (4 \times 7 - 6 \times 3) = 10$ i.e. $|A| \neq 0$

The above non-zero result of the determinant A shows that there exists an inverse in the matrix. Thus, proceeding further, and computing the cofactors of A we get.

$C^{11} = 7$, $C^{12} = -6$, $C^{21} = -3$ and $C_22 = 4$

$$[CF] = \begin{bmatrix} 7 & -6 \\ -3 & 4 \end{bmatrix}, \therefore \text{Adj. } A = [CF]^t = \begin{bmatrix} 7 & -6 \\ -3 & 4 \end{bmatrix}^t = \begin{bmatrix} 7 & -3 \\ -6 & 4 \end{bmatrix}$$

Thus, $$A^{-1} = \frac{1}{10}\begin{bmatrix} 7 & -3 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 \\ -0.6 & 0.4 \end{bmatrix}$$

We have, for solution

$$X = A^{-1}.B = \begin{bmatrix} 0.7 & -0.3 \\ -0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 8 \\ 17 \end{bmatrix} = \begin{bmatrix} 0.7 \times 8 + (-0.3 \times 17) \\ (-0.6 \times 8 + 0.4 \times 17) \end{bmatrix} = \begin{bmatrix} 5.60 - 5.10 \\ -4.80 + 6.80 \end{bmatrix} = \begin{bmatrix} 0.50 \\ 2.0 \end{bmatrix}$$

Thus, we have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.50 \\ 2.0 \end{bmatrix} \Rightarrow x = 0.50$ and $y = 2.00$

1.20 SUMMARY

The process of selecting things is called combinations and that of arranging the selected things is called counting permutations. Permutation and combination provide the rules of the different numbers in a wide variety of problems relating to statistics & quantitative techniques.

Notes

A matrix may be defined as an orderly arrangement of some numbers in certain rows and columns enclosed by some brackets, subscripted by magnitude of its order and denominated by some capital letters.

The term determinant can be defined as "A numerical value obtained from a square matrices of the co-efficient of certain unknown variables by two bars by the process of diagonal expansion to tell upon a given algebraic system.

1.21 SELF ASSESSMENT QUESTIONS / PROBLEMS

1. In how many ways can 5 persons be seated around a table so that none of them is adjacent to his neighbour?
2. After publication of a supplementary result 4 students have applied for 5 hotels. In how many ways can they be accommodated in the hostels?
3. Find the number of ways in which the kings of a pack of playing cards can be arranged in a row?
4. In how many ways can 7 students be accommodated in 3 hostels?
5. Find the number of permutations that can be made from the word "amalgamation".
6. Find the number of arrangements of the letters of the word MATHEMATICS. In how many of these vowels occur together?
7. In how many ways can 8 flowers of different colours be strung into a garland.
8. In how many ways can 5 red and 4 while balls be drawn from a bag containing 10 red and 8 white balls?
9. In how many different ways selection of 4 books can be made from 10 different books if two particular books are always selected?
10. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can be the selections be made.
11. Determine the minor and cofactors of the following determinant

$$|B| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}_{3x3}$$

12. Find the value of the determinant given below using the minor and cofactor expansion method:-

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}$$

13. Find the value of the determinant |B| from the following using the minor and cofactor expansion method.

$$|B| = \begin{vmatrix} 2 & -5 & 3 \\ 3 & -1 & 2 \\ 1 & -2 & 1 \end{vmatrix}$$

14. Using the relevant property, prove that

$$\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$$

Notes

15. Solve the following linear equations with the help of Cramer's Rule

    x + 2y + 3z = 6

    2x + y + z = 4

    x + y + 2z = 4

16. Divide the matrix A by the matrix B where

    $$|A| = \begin{bmatrix} 4 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\,x3} \text{ and } |B| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

17. Find the determinant of the following matrix:

    $$|C| = \begin{bmatrix} 2 & -5 & 3 \\ 3 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

18. Find the inverse of the following matrix.

    $$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$$

19. Using the technique of matrix algebra solve the following simultaneous equations:

    4x + 3y = 8

    6x + 7y = 17

Notes

UNIT 2 MEANING AND CLASSIFICATIONS OF QUANTITATIVE TECHNIQUES

Structure

2.0 OBJECTIVES

After going through this unit you will be able to understand:

- quantitative techniques for decision making
- use of decision making in various field of business
- classification of quantitative techniques
- meaning and scope of statistics
- statistical data and their presentation in useful form
- measures of central tendency
- measures of dispersion
- how to use the data for solving practical problems

2.1 QUANTITATIVE TECHNIQUES

Introduction of QT

Managerial activities have become complex and it is necessary to make right decisions to avoid heavy losses. Whether it is a manufacturing unit, or a service organisation, the resources have to be utilised to its maximum in an efficient manner. The future is clouded with uncertainty and fast changing and decision making – a crucial activity. In such situation there is a greater need for applying scientific methods to decision making to increase the probability of coming up with a good decisions. Quantitative technique is a scientific approach to managerial decision –making. The successful use of Quantitative Technique for management would help the organisation in solving complex problems on time, with greater accuracy and in the most economical way.

Meaning of QT

Quantitative technique is the scientific way to managerial decision-making, while emotion and guess work are not part of the scientific management approach. This approach starts with data. Like raw material for a factory, this data is manipulated or processed into information that is valuable to people making decision. This processing and manipulating of raw data into meaningful information is the heart of scientific management analysis.

Classification of Quantitative Techniques

QT model can be classified in the following categories:

1. ***Linear programming Models:*** When decision making pertains to profits, cost etc. and these parameters have a linear relationship of several variables, the model is known as Linear Programming Model having constraints or limitations on various resources also as linear function of the decision variables or parameters.

2. ***Sequencing Models:*** Instead of assigning the jobs in a definite activity system, when we have to determine in what sequence the activities should be performed out of given resources in the most cost\time effective manner, the models are called Sequencing Models.

3. ***Waiting Line or queuing Models:*** These models are used to establish a trade-off between the cost waiting of customer and that of providing service following a queue system. In this case, we have to describe various components of the system such as traffic intensity, average waiting time of the customer in the queue, average queue length, etc.

4. ***Games Models:*** These models are formulated and utilised to describe the behaviour of two or more opponents or players who are performing the functions to achieve certain objectives or goals and in the bargain, would gain or loose in the business process. Such models are very effectively used for optimising strategies of the players with respect to anticipated strategies of the competing players.

5. ***Dynamic programming Models:*** These models are the offshoots of the mathematical programming for optimising the multistage decision processes. The problems are solved by first dividing the problem into sub-problems or stages and solving them sequentially till the original problem has been solved.

6. ***Inventory models:*** These models are primarily meant for working out optimal level of stocking and ordering of items for a given situation. Main objective is to optimise the cost under conflicting requirements of ordering, holding and shortages.

7. ***Replacement models:*** These models are utilised when we have to decide the replacement policy for an equipment for one reason or the other. The deterioration of efficiency of the equipment with use and time is the reason for such replacement whether partial or full.

8. ***Simulation models:*** these models are utilised when we want to evaluate the merits of alternate course of action by experimenting with a mathematical mode of the problem and the variables in the problems are random. Thus repetitions of the process by using simulation models provide an indication of the merit of the alternate course of action with respect to the decision variables.

9. ***Network models:*** these are basically project management models utilised in planning, monitoring and controlling various projects where utilisation of human and non-human resources has to be optimised with reference to the time and cost available for the project. CPM\PERT as basic network model help in identification of important bottleneck or potential trouble areas.

10. ***Decision analysis model:*** these models are used for selection of optimal strategy of operation given the possible payoffs and their associated probability of occurrence. The models are used for decision making process under uncertainty or risk conditions.

Notes

Scope of Quantitative Techniques

The scope and areas of application of scientific management are very wide in engineering and management studies. Today there are a number of quantitative software packages are available to solve the problem using computers.

This helps the analyst and researchers to take accurate and timely decisions .This book is brought out with computer based problem solving. A few specific areas are given below:

1. ***Finance and Accounting:*** Cash flow analysis, capital budgeting, financial planning, dividend and portfolio management.
2. ***Marketing Management:*** Selection of product mix, sales resources allocation and assignment, market research decision, pricing and competitive decision..
3. ***Production Management:*** Facilitates planning, manufacturing, aggregate planning, inventory control, quality control, work scheduling, mentainance and project planning and scheduling, job sequencing.
4. ***Personnel management:*** Man power planning, resource allocation, staffing, scheduling of training programs, recruitment policy and job evaluation.
5. ***General management:*** Decision support systems and management of information systems, organisational design and control, software process management and knowledge management.
6. ***Research and development:*** Determination of areas of thrust for research and development, selection criteria for specific project, analysis for alternative design and reliability.
7. ***Defence:*** Optimum level of force deployment, optimum weaponry systems transportation cost, assignment suitability.

2.2 STATISTICS

Meaning

The word, ‘statistics’ has been derived from the Latin Word, ‘Status’ which means a political state. This word also resembles with the Italian word ‘Statista’, German word ‘Statistik’ and French word ‘Statistique’ carrying the same meaning of a state. The word was for the first time introduced by Professor Gottfried Achenwall (1719-1772) and J.F. Von Briedfield (1770) in their famous work, “Elements of Universal Erudition”.

The term statistics is used in two different senses-plural & singular. In plural sense, it refers to a set of numerical data collected & presented in a systematic manner in order to fulfil certain objectives. In singular sense, it refers to a branch of knowledge which has some principles, rules & methods. In short, it refers to statistical methods.

Definition

Statistics is the science which deals with collection, presentation, analysis and interpretation of numerical data.

–Bowley, Croxton and Cowden

By statistics we mean quantitative data affected to a marked extent by multiplicity of causes.

– Yule and Kendall

By statistics we mean aggregate of facts affected to a marked extent by multiplicity of causes, numerically expressed, enumerated or estimated according to reasonable standard of accuracy, collected in a systematic manner for a predetermined purpose and placed in relation to each other.

– Prof. Horace Secrist

Notes

There are various characteristics of statistics:-

1. ***Aggregate of facts:*** Single or an isolated figure can't be termed as statistics because such figure lacks comparability. Thus, a single figure relating to mark, production, birth, death etc. dose not constitute statistics.

2. ***Affected to a marked extent by multiplicity of causes:*** Statistics is affected by a large variety of factors operating together.

   For example: the production of rice is affected combinedly by rainfall, fertility of soil, quality of seeds & method of cultivation. As such it is not possible to study separately the effect of each of these forces on production.

3. ***Numerically expressed:*** All statistics are expressed in number. It means only numerically data constitute statistics. The qualitative statements such as performance of the students has increased, production of rice are not sufficient, etc; do not form statistics. For this reason, the qualitative characteristics such as beauty, honesty, etc which can't be expressed numerically do not constitute statistics. But, when they are expressed by giving ranks or marks, they can be termed as statistics.

4. ***Enumerated or Estimated:*** There are two ways of deriving figures relating to a particular phenomenon actual counting & estimation. Even though enumeration is precise & accurate, still in certain cases we prefer estimation because, either the scope of enquiry is very vast or greater accuracy is not necessary.

5. ***Collected in a systematic manner:*** The collection of data should be systematic. For this, proper planning & deployment of trained personnel is essential. If the data are collected haphazardly, it may lead to erroneous conclusion.

6. ***Pre-determined purpose:*** The purpose of collecting data must be clearly spelt out well in advance. The purpose should be specific & well defined. If the purpose is unclear or ambiguous it may lead to wrong conclusion.

   For example: our purpose is to collect data on retail price. If we set our objective to collect prices, one may collect retail price of some articles & wholesale price of some other article. As a result all our efforts will be wasteful & conclusion will be wrong.

7. ***Placed in relation to each other:*** The basic purpose of collecting data is to make comparison. In order to make genuine comparison, the data should be homogeneous- related to the same subject matter.

   For example: population of India may be compared with the population of other countries. But if the mark of a student & his parent's income are placed together, it does not make any sense & hence cannot be termed as statistics.

Scope of Statistics

Scope refers to the area where statistics can be used. Now-a-days almost in every field statistics is applied & therefore its scope is so vast that it can not be defined. However, the application of statistics in some important areas is given below:-

1. ***Statistics & the state:*** In the earlier days the rulers relied heavily on staistics for framing suitable military & fiscal policy. They collected data on population, revenue & taxes for smooth management of the state. The reliance on statistics by a state has now become more as it has become a welfare state. It collects data on production, consumption, population, defence, taxes, agriculture, transportation, etc. & analyses these data to frame policies for future. Govt. is the biggest collector & user of statistical data.

2. ***Statistics & Economics:*** Improvement of statistical methods such as sampling, probability, correlation & index number has made statistics & economics close friends. Data on consumption helps us to get an idea how people of different strata spend their income. Data on production enables us to know whether our production & productivity match with other countries.

Notes

An economist studies demand and supply, cost of production, prices, competition, revenues and expenditure of a state, implication of the reforms etc, with the help of statistics and consequently helps in solving different economic problems.

3. ***Statistics & Business:*** Statistical methods are widely used in business & trade for production & financial analysis, distribution, planning, market, research & forecasting. A business indeed runs on estimates & profitability. A manager has to plan, organise, supervise & control the operations of a business amidst uncertainty. The degree of uncertainty can be reduced & rational decision can be taken by collecting & analysing past records.

4. ***Statistics & Research:*** Research work cannot be undertaken without the help of statistics. Statistics is indispensable for research work. By analysing the collected data we can invent new product & new method of production & marketing.

   For example- by conducting research on crop yield by using different fertiliser, we can formulate ways to increase yield.

5. ***Statistics & Astronomy:*** The astronomers collect data on the movement of heavenly bodies, analyse those & draw inferences. They apply statistical methods to go deep into their study. They take a lot of measurements & apply the method of least square to arrive at a conclusion. Thus, astronomers will be handicapped without statistics.

6. ***Statistics & Mathematics:*** Statistics & mathematics are inter-dependent. With the development of mathematical statistics these two subjects have become closer. Statistics takes several theories of mathematics to develop different statistical methods. A simple knowledge on mathematics helps to understand different statistical methods such as average, dispersion, correlation etc.

7. ***Statistics & Biology:*** The authenticity of any biological statement can only be verified with the help of statistics.

   For example, tall fathers have tall son's can only be proved by taking sample & by measuring the height of both father & sons.

8. ***Statistics & Natural Sciences:*** The natural sciences such as physics, chemistry, geology, engineering, medicine, botany, zoology etc. heavily depend on statistics.

   For example, a doctor needs various statistics relating to temperature, pulse rate, blood pressure etc. before treating a particular patient. Similarly, in physics & chemistry we undertake experiments, record the data & draw conclusion.

9. ***Statistics & Education:*** Statistics is extensively used in the field of education. Various statistics are necessary to adopt new course, to increase the strength of student, to compare performance of students in different years, to evaluate faculty position, etc. Even a good number of scholars are engaged in social research in different educational institutional for which they need statistics relating to their field of research.

10. ***Statistics & Common man:*** Most of the people make use of statistical methods while taking decision either knowingly or unconsciously.

    For example, before buying a refrigerator we are interested to know the price of different brands to have an idea about the price range & the average price. A farmer, before raising a crop in a particular field, uses the idea of correlation of crops yield with season.

11. ***Other users:***

    a. *Banker:* A bank needs data on consumers, growth of deposit, growth of borrowing, deposit credit ratio, bad debt etc., to frame strategy for future.

    b. *Insurance companies:* Insurance companies need data relating to life table, premium rate, number of different policies, claim paid, etc. for making policy decision.

    c. *Stock broker:* Statistics is equally important for broker, speculator & investors. They study the trend of share prices by analysing past data & accordingly undertake their transaction.

d. *Public utility service:* Public utilities such as railway, waterworks and electricity companies make use of statistical data for smooth management of their affairs.

Limitations

Despite various functions and utilities, statistics suffers from the following limitations:

1. ***Ignores individual items:*** An isolated or single figure however important may be, does not come within the ambit of statistics because statistics, refers to a group of data.

   For example: the population of India is 110 crores is not a statistical statement.

2. ***Considers the data which can only be expressed numerically:*** Statistics does not take into account the data which can't be expressed quantitatively.

   For example: intelligence, beauty, honesty, etc. cannot find a place in statistics unless they are converted to quantitative measurement by assigning certain marks or ranks.

3. ***Statistical laws are true only on averages:*** Statistical laws are not as perfect as the laws of physical science. They are true only on an average basis because the statistical figures are affected by a number of causes & it is not possible to study the effect of each of these forces separately. As such the conclusions arrived at are not perfectly accurate & the result is true under certain conditions.

4. ***Does not reveal the entire story:*** Statistics cannot reveal the entire picture of a problem. A problem is affected by a number of causes & many of its causes lack statistical approach. Therefore, it fails to disclose all the truth.

5. ***Likely to be misused:*** Statistics, if not used by expert or experienced person, is likely to be misused. A great deal of skill & experience is necessary to draw valid conclusion. It may lead to fallacious conclusion in the hands of inexperienced person. Further, proper understanding of the subject is essential for correct interpretation.

6. ***Sometimes statistics gives strange results:*** Statistics sometimes gives absurd results.

   For example: In a study 50 families, the number of children per family may be 1.7 which is quite absurd.

Functions of Statistics

Various functions of statistics are explained under the following headings:

1. ***Simplifies & condenses mass data:*** Statistics collects data & simplifies them by presenting in suitable form. A large mass of complete data is condensed briefly with tables, charts & graphs so that they can be readily understood & analysed. Thus, without the help of statistics, it would be very difficult to study & analyse a mass of raw data to draw valid conclusion.

2. ***Presents facts in a definite form:*** Statement of facts, if conveyed in exact quantitative term, are more convincing. Statistics does it by converting the general statements into definite quantitative form.

3. ***Facilitates comparison:*** The significance of one data cannot be judged unless it is compared with another similar data. Statistics helps in comparison of two sets of data by applying the techniques of average, standard deviation, co-efficient of variation etc. For example:- if the average marks in CHSE of Gopabandhu Science College and Maharshi college are put side by side, it becomes more meaningful.

4. ***Formulation & testing of hypothesis:*** Hypothesis is a tentative proposition formulated for empirical testing. In simple words, it is a tentative answer to a research question. It is tentative answer because its veracity can be evaluated only after it has been tested empirically. We make take a hypothesis that a particular injection is effective against dengue. This statement is called a hypothesis. Statistics not only helps to form hypothesis but also applies techniques to evaluate the truthness of the statements.

Notes

5. ***Guides in formulation policies:*** Statistics provide basic ingredients for suitable policy formulation. Statistical data on population, age, education, etc. helps the government in framing policy on social & economical matters. It would be difficult, rather impossible, to formulate policies without adequate facts & figures.

6. ***Forecasting:*** Statistics, by studying the trend of the data of past years, can predict the future occurrence. An automobile company can estimate the production of automobile for the next year by analysing the sale of past years. Statistics helps in forecasting through various techniques of time series, regression analysis, extrapolation, etc.

7. ***Establishes relationship:*** The relationship between two or more variables can be established with the help of statistics. The relationship between demand & supply, production & consumption, cost & profit, etc. can be measured by statistics with the help of correlation analysis & regression co-efficient.

8. ***Draw valid conclusion:*** Ultimate aim of every statistical enquiry is to draw a valid conclusion. This is possible because statistics collects data systematically & present them in suitable form. Consequently, it makes the researcher's task easy for analysis & interpretation of data. Therefore, rational & valid conclusion can be drawn.

9. ***Reduces uncertainty:*** The complexity of business, globalisation & the intensity of competition has made the economic environment more uncertain. A business manager has no option but to operate in this volatile environment. Statistics helps to reduce such uncertainty through the application of various tools such as the theory of probability, linear programming, etc.

10. ***Enlarge & enrich human knowledge:*** Analysis of data & the outcome of the interpretation have enlarged the domain of human knowledge. Apart from these, the techniques of statistics has enriched human mind. Many fields would have remained unearthed had there been no statistics. Furthermore, statistics brings objectivity to the study.

2.3 STATISTICAL DATA COLLECTION

Data constitutes the foundation of statistical analysis. The data collected through statistical investigation are in an unorganized form and are termed as raw data. These unorganized masses of figures are voluminous, unwieldy and un-comprehensible and therefore, these are to be edited, processed and organized for making useful analysis and interpretation.

According to Crum, Patton and Tebbutt, "Collection means the assembling, for the purpose of particularly investigation of entirely new data, presumably not already available in published sources".

Statistical data are of two types: Primary data and Secondary data.

Primary Data

The data collected for the first time by an individual or an organisation for their own use from the source of origin are termed as primary data. For ex- the census data collected by the government. Primary data are authentic and original in character and are generally obtained through surveys.

Broadly speaking there are five different methods of collecting primary data which are as under:-

I. Direct personal investigation.

II. Indirect oral investigation

III. Through local correspondents

IV. Mailed questionnaire.

V. Schedules sent through enumerators.

I. Direct Personal Investigation

Notes

Under this method, the investigator himself personally goes to the source of the data & collects the necessary information either through interview with the informants or through observation of the data accuracy on the spot. This method is suitable particularly where intensive study of the phenomenon is required.

Merits:

1. The data collected under this method is not only reliable but also correct.
2. Personal contact gives encouragement to the informants. So they respond positively.
3. Sensitive questions can be twisted to get response.
4. Supplementary information about the respondent & environment can be obtained.

Demerits:

1. It is not only time consuming but also involves more cost.
2. The personal prejudices of the investigator may influence the answer of the informants which may prove disastrous.
3. Employment of untrained interviewer may defeat the very purpose of data collection.

II. Indirect Oral Investigation

Under this method, the investigator contacts a third person who is capable of supplying information about the informant. The third party called witness is expected to know all the details about the informant. This method is followed where the information to be obtained is sensitive & the informants are not likely to provide an answer.

Merits:

1. A wide area can be covered within a given time.
2. It needs less amount of resources in terms of time, energy and money.
3. Prejudices of the original informants are eliminated as the information are recorded from the disinterested third parties.

Demerits:

1. The facts obtained from the third party may not be reliable at times.
2. The third parties, may at time be actuated by some motive & thus depose fabricated information.
3. A wrong & improper choice of the witness may vitiate the result of the enquiry.

III. Through Local Correspondents

Under the method, the investigator collects the required data through the local correspondents & agents placed in the different regions of the country. Such type of data collecting method is usually adopted by the newspapers, or periodical agencies & various departments of a govt., who require regular information from a wide area on various matters viz. Economics, Commerce, Politics, Agriculture, Sports, Accidents, Riots, Strikes, Lock-outs, Stock market, Birth & Death etc. The correspondents, or the agents so appointed at the different localities collect the relevant information in their own ways & fashions and submit them periodically to the investigating offices for their necessary use & analysis.

Merits:

1. It does not require any formal procedure & hence, a lot of botheration associated therewith is avoided.
2. It is less expensive in terms of money, time & energy.
3. The data can be collected expeditiously from a wide area.

Notes

Demerits:

1. The data are not very reliable as they are obtained informally through the correspondents who collect the data in their own & according to their own liking & decision.
2. The local agents may use foul play in supplying the data regularly & correctly.
3. The data are likely to be fabricated & twisted by the correspondents to aggrandise their ulterior motives.

IV. Mailed Questionnaires

Under this method, a list of relating questions to the problem under study is set & the same is sent along with a covering letter to the selected informants by mail i.e. by post who are requested to furnish the necessary data therewith by return of post. This method of data collection is usually followed by private bodies, individuals, research scholars, institutions & govt. as well.

Merits:

1. It ensures a reasonable standard of accuracy when the investigation is properly conducted.
2. The information gathered under this method is original & so, more authentic.
3. The chief advantage of this method is that it is the most economical method in terms of time, money & energy, provided the respondents respond to the questionnaire in time.

Demerits:

1. This method cannot be used, if the informants are illiterates & indifferent.
2. This method is not flexible i.e. adjustable to the changing circumstances.
3. In case of inadequate or incomplete answers it may be difficult to get the correct information.

V. Schedules Sent Through Enumerators

Under this method, schedules are sent to the informants through the enumerators who read out the questions from the schedules to the informants & record their answers on the same schedules. Before doing so, the enumerator explains first the aims & objectives of the enquiry to the informants & seeks their co-operation in the recording of the data.

Merits:

1. The data collected prove to be more reliable & dependent, since they are recorded by the enumerators trained in the matter.
2. It ensures large response from the respondents since they are called on personally by the enumerators.
3. It is possible to rectify the erratic replies given by some cunning informants through cross questions being put by the enumerators on the spot.

Demerits:

1. This method is extremely expensive as it requires an army of trained enumerators to be sent along with the schedules. Hence, this method can be afforded only by the big organisations.
2. This method is extremely time taking in nature as it requires the enumerators to approach each of the informants at their door-step & explain everything to them in clear terms.

Secondary Data

Secondary data are the data which are collected from some secondary sources i.e. the source of reservation or storage where the data are stored after being collected & used for some purpose by some agency. These data are available either in the published or unpublished form & are collected for the time other than the first time by an investigator or agency for any subsequent investigation. For collecting the secondary data, the investigator has to contact the relevant sources of the secondary data in either of the following methods.

Notes

1. Personal visit to the source.
2. Correspondence with the authorities of the source.
3. Subscription to the periodicals.

1. ***Personal visit to the source:*** Under this method, the investigator personally visits the source of the information viz. libraries, museums, govt. offices, private & semi-govt. institutions, private persons like research scholar, MLA's, MP's etc. & requests the authority concerned to provide him with the necessary records & permit him to take down the required data there from. This method is comparatively more expensive & arduous in nature. But, it takes a little time & gives more accurate & dependable results.
2. ***Correspondence with the authorities of the sources:*** Under this method, the investigator makes correspondence with the authorities of the relevant sources for getting the desired data by post or through some special messengers. For this, he writes very polite letters & reminders to the authorities concerned to supply him the necessary data for his study. In this method, the data are collected usually in the form of reports, circulars, bulletins, statutes etc.
3. ***Subscription to the periodicals:*** Under this method, the investigator subscribes to certain periodical, regularly to obtain the relevant data there from. This method of obtaining the secondary data is, no doubt, very comfortable but all the same it is very much time taking & expensive. Moreover, the investigator may not get all the information from the periodicals to which he may subscribe.

Sources

Secondary data may be either in published or unpublished form.

1. Published Sources

A. Report & official publication of the govt. and other organisation. It includes:
   i. Internal organisation such as International Monetary Fund(IMF), World Bank, International Labour Organisation, United Nations etc.
   ii. National Organisations such as Central Statistical Organisation (CSO), National Sample Survey Organisation (NSSO), Registrar General & Census Commission of India, Reserve Bank of India, Indian Statistical Institute, Indian Council of Agricultural Research, etc.
   iii. State level organisations such as Economic Survey of Odisha, Directorate of Statistics, Directorate of Co-operatives etc.

B. ***Journal & magazines:*** It includes international, national & state level journals & magazines such as Indian Journal of Commerce, Odisha Journal of Commerce, Indian Journal of Economics, Journal of the Institute of Chartered Accountants, Sports Journal etc.

C. ***Newspapers:*** Articles in different local & national level newspaper provide a lot of latest data which can be used by the researcher.

D. Semi- Official publication of various local bodies such as district board, Notified Area Council, Municipality, Municipal Corporations, Development Trusts etc are important sources of secondary data.

E. ***Annual reports.*** Every company publishes its annual report at the end of the year regarding the progress made in the previous year. At the same time, data relating to past 3 years or 5 years are also attached to it for the purpose of comparison. The data collected from annual reports is an important source of secondary data to the researcher & financial analyst.

2. Unpublished Source

I. Record maintained by the private firms.

II. Material retained by the research scholars.

   Documents relating to registration, permit licenses, loan etc.

Notes

IV. Confidential records maintained by various departments of Govts, Chambers of Commerce, trade associations, trade unions, labour bureaux etc.

V. Administrative records relating to internal activities of an organisation viz. cost records, final accounts, progress reports, Operational budgets, performance records etc.

2.4 CLASSIFICATION OF DATA

After collection and edition of data, the next step is classification of data. Classification means arrangement of data into various groups or categories of homogenous character. Without classification, a heap of data will look haphazard and confusing and no purposeful meaning can be interpreted from them.

According to Secrist, "Classification of data is a process of arranging data into sequences and groups according to their common characteristics or separating them into different but related parts".

In other words, classification is the process of arranging or rearranging the data into certain groups, or classes according to their resemblances or similarities on a certain point viz., age, religion, education, income, expenditure or occupation, etc.

2.5 TABULATION AND PRESENTATION OF STATISTICAL DATA

Tabulation means a symmetric arrangement of statistical data in columns and rows. It is done for making meaningful analysis, comparison and interpretation of the data collected and classified.

According to Horace Secrist, "tables are a means of recording in permanent form the analysis that is made through classification & by placing in juxtaposition things that are similar & should be compared".

Even though the number of parts of a table depends upon the nature of the data still, every table should have the following general parts.

1. ***Table number:*** Every table should bear a distinctive number for identification & future reference. The table number should be placed at the middle of the table just above the title.
2. ***Title of the table:*** Title is a description of the contents of a table. The title should convey the nature of data, period & area covered. The title should be short, self explanatory & complete & should be written either in bold or capital letters.
3. ***Caption (column heading):*** When a table contains a number of columns each column should be given a title. The unit of measurement must be clearly mentioned. The caption should be clearly defined & must be placed in the middle of the column.
4. ***Stub (row heading):*** Each row of a table must have a heading which is termed as stub. The row heading should be condensed without sacrificing precision clarity, so that it can be accommodated in a single row. The stub must explain the data given in a row.
5. ***Body of the table:*** It is the vital & largest part of the table which contains numerical information. The data should be arranged according to the description given for each row & column.
6. ***Totals:*** The figures of each of the columns and sub columns, and that each of the rows should be totalled. For getting the totals of the different columns in a prominent manner, a separate 'row of totals' is provided at the bottom of the table. Similarly, for getting the totals of the different rows in a prominent manner, a separate 'column of totals' is provided at the right end of the table.
7. ***Unit of the measurement:*** The unit in which data are expressed must be stated in the right top corner just below the title of the table. For example: - "in crores", "number in thousand" etc., should be mentioned preferably within brackets. It is also known as head note or prefactory note.

Notes

8. ***Source of data:*** A note at the bottom of each table must be given to indicate the source from which data have been brought.

9. ***Footnotes:*** Statements of explanations given at the bottom of the table are called footnotes. It is an explanation given for some specific items which the reader fails to understand from the title, caption or stubs. It also points out the reasons of abnormal data.

Presentation of Data

Since raw data does not speak full volume of information for effective utilisation, the presentation of it in the concise, coherent and structured form becomes imperative. The incomprehensibility of the raw data can be overcome by putting these into meaningful manner so that it can be analysed to be of use. Some of the ways of classification and presentation of the data are described under:

1. The data array is the simplest method of arranging the data. The given data can either be arranged in ascending order or in descending order. By arranging the data in this manner, we can quickly notice the trend and spread of the data i.e., the highest and lowest value and steps of variation.

2. When the data is large, we can arrange it in the form of tables or can be drawn into a graphical form to summarize the utility of the data. The table indicates the logical presentation of data with reference to some other variable on terms of its relation to time.

3. A graph is a sort of chart through which statistical data are represented in the form of lines or curves drawn across the coordinated points plotted on its surface. It helps us in studying the cause and effect relationship between two variables. This implies that with the help of graphs we can measure the extent of a change in one variable when another variable changes by a certain amount. Graphs are useful for studying both time series and frequency distributions.

   Some of the important graphs are historigram, range chart, graphs of frequency distribution, histograms, frequency polygon, and ogive.

   Historigram: When a graph represents data relating to a time series, it is called a graph of time series or historigram.

   Range Chart: It is a type of graph which is drawn on a natural scale to highlight the range of variations in the values of a particular variable with reference to different periods or times.

   Graphs of Frequency Distribution: A graph which represents the frequencies of the different values of a variable is called a frequency graph or a graph of frequency distribution.

   Histograms: This is a type of non-cumulative frequency graph drawn on a natural scale in which the respective frequencies of the different classes of values are represented through vertical rectangles drawn closed to each other. With the help of this graph we can easily determine the value of mode, a measure of central tendency.

   Frequency Polygon: This is a frequency graph which is drawn as an improvement over the histogram. This graph is drawn in the form of a smooth curve by joining the mid-points on the top of all the rectangular bars of a histogram.

   Ogive (cumulative Frequency Polygon): These are cumulative frequency graphs drawn on a natural scale to determine on a natural scale to determine the values of certain factors viz., median, quartiles, deciles, percentiles. In these graphs, the class limits are shown along the X-axis and the cumulative frequencies are shown along the Y-axis. There are two types of ogives, less than ogive (frequencies arranged in less than order) and more than ogive (frequencies arranged in more than order). The less than ogive curve is obtained by plotting the cumulative frequencies of ‘less than type’ against the upper boundary of the class interval. It gives an increasing curve from left to right and takes the shape of an elongated S. The more than ogive curve is obtained by plotting the ‘more than frequencies’ against the lower boundaries of the class interval. The curve obtained is a gradually decreasing curve sloping downwards from right to the left and takes the shape of an elongated S upside down.

Notes

4. ***Diagrams:*** Diagrams are used by statisticians to present the data before the common mass, who do not understand the numerical figures and for whom the numerical figures are considered very much boring and complicated. They take the form of geometrical figures viz; points, lines, bars, squares, rectangles, circles, pies, cubes, pictures, maps, charts, etc. There are different types of diagram which can be broadly classified into three categories:

   (a) *Pictograms:* It refers to the pictures or cartoons. In this case, appropriate pictures are drawn to represent the quantitative data relating to a phenomenon.

   (b) *Cartogram:* It refers to a map through which statistical information are represented in different manner viz., shades, dots, pictograms, columns. The regional distribution of data viz., distribution of rainfall in various parts of a country, deposits of minerals in various regions, density of population in various geographical areas etc., are well represented through this type of diagram.

   (c) *Dimensional diagram:* A diagram in which certain dimensions are displayed in a prominent and proportional manner is called a dimensional diagram. These are of three types: a) one dimensional diagram (a diagram in which the size of only one dimension i.e., length is fixed in proportion to the value of the data). These are also popularly known as bar diagrams. b) two dimensional diagram (a diagram in which two dimensions viz., length and width are highlighted in proportion to the value of the data). The most popular diagrams that come under this category are rectangles, squares and circles or pies.

   In pie-diagrams, the different segments of a circle show percentage contribution of various constituents to its total picture. This sub-divided circle diagram is called an angular or pie-diagram.

2.6 CHARACTERISTICS OF FREQUENCY DISTRIBUTION

Frequency is defined as the number of times a particular item occurs in a given set of data. Arrangement of data into classes and mentioning the number of observations in each class is called a frequency distribution. It presents the data in more compact form.

The next step after the completion of data collection is to organize the data into a meaningful form so that a trend, if any, emerging out of the data can be seen easily. One of the common methods for organizing data is to construct frequency distribution. Frequency distribution is an organized tabulation/graphical representation of the number of individuals in each category on the scale of measurement. It allows the researcher to have a glance at the entire data conveniently. It shows whether the observations are high or low and also whether they are concentrated in one area or spread out across the entire scale. Thus, frequency distribution presents a picture of how the individual observations are distributed in the measurement scale.

There are four important characteristics of frequency distribution. They are as follows:

- Measures of central tendency and location (mean, median, mode)
- Measures of dispersion (range, variance, standard deviation)
- The extent of symmetry/asymmetry (skewness)
- The flatness or peakedness (kurtosis).

2.7 MEASURES OF CENTRAL TENDENCY

The word ‘measures’ means methods & the ‘central tendency’ mean average value of any statistical series. Thus, the combined term ‘measures of central tendency’ means the methods of finding out the central value or average value of a statistical series or any series of quantitative information.

Characteristics of an Ideal Average

There are various characteristics of an average

1. It should be rigidly defined, simple to understand and calculate.
2. It should be based on all the observations of the series.
3. It should be capable of further algebraic treatment.
4. It should not be greatly affected by the values of extreme items of a series.
5. It should not be affected by fluctuations in sampling.

MEAN

Arithmetic Average

An arithmetic average may be defined as the quotient obtained by dividing the total of the values of a variable by the total number of their observation or items. The fundamental formula for its calculation is given by:

$$\overline{X} = \frac{\sum x}{N}$$

where $\overline{X}$ = arithmetic average, x = value of a variable

Σx = Sum of the values of a variable and N = total number of observations or items.

Short-cut Method

Under this method, a value preferably from the middle is first assumed to be the value of the arithmetic average. Then, from the assumed average, the deviations of the different items of the series are found out. The average of such deviations is then added to the assumed average.

The resultant figure comes out to be the value of the arithmetic average.

$$\overline{X} = A + \frac{\sum Fd}{N}$$

where A = assumed average

d = deviation of the item from the assumed average, i.e., (X-A)

Σ d = sum of the deviations from the assumed average

and Σ Fd = sum of the products of deviations and their corresponding frequencies.

This method is suitable when the series is made of big and complicated values or numbers.

Step Deviation Method

When the figures of deviations appear to be big and divisible by a common factor, this method should be applied to compute the mean at an ease. Under this method, the figures of deviations are reduced by dividing them all by a common factor. The formula is given as under:

$$\overline{X} = A + \frac{\sum Fd'}{N} \times c$$

where c = common factor by which each of the deviations is divided

and d′ = the deviation from the assumed average divided by the common factor i.e., $\left(\frac{X-A}{c}\right)$

Example: From the following data relating to the monthly salaries of the teaching staff of a college determine the average salary of a by all the possible methods.

Salary Rs.	2,200	2,500	3,000	3,700	4,500
No. of Staff	5	10	15	7	3

Notes

Solution:

Computation of arithmetic average of salary of the staff of a college

I. Direct Method

Salaries Rs. X	**No. of Staff F**	**FX**	
2,200	5	11,000	
2,500	10	25,000	
3,000	15	45,000	
3,700	7	25,900	
4,500	3	13,500	
Total	N = 40	1,20,400	N =40

We have, $\overline{X} = \frac{\sum FX}{N} = \frac{1,20,400}{40} = 3,010$

Hence, the average of monthly salaries of the staff is Rs. 3010.

II. By the Short-cut method (where A=3000)

X	F	(X–A) = d	Fd
2,200	5	-800	-4,000
2,500	10	-500	-5,000
3,000	15	000	0000
3,700	7	700	4,900
4,500	3	1,500	4,500
Total	N = 40	—	ΣFd = 400

We have, $\overline{X} = A + \frac{\sum Fd}{N} = 3000 + \frac{400}{40} = 3000 + 10 = 3010$

III. By Step deviation method (where A =3000 and c = 100)

X	F	(X-A)=d	(d/c)=**d′**	**Fd′**
2,200	5	-800	-8	-40
2,500	10	-500	-5	-50
3,000	15	000	0	00
3,700	7	700	7	49
4,500	3	1,500	15	45
Total	N = 40	-		$\Sigma Fd' = 4$

We have, $\overline{X} = A + \frac{\sum Fd'}{N} \times c = 3,000 + \frac{4}{40} \times 100 = 3010.$

Merits

1. It is rigidly defined.
2. It is easy to understand & simple to calculate. Thus, it is a popular or common men's average.
3. It is capable of further algebraic treatment & thus, in almost all the advanced studies of statistics like measures of dispersion, skewness, correlation, sampling etc, it is taken as a formidable factor.

4. It has maximum mathematical properties & thus, the various factors involved in its formula can be computed easily under various methods viz. direct, short-cut, step deviation & shortest method.

Demerits

1. It is greatly affected by the values of extreme items of a series & thus it fails to give a representative value of the series.
2. It cannot be calculated in a distribution with open end classes without making assumption on the size of the class intervals of the open end classes.
3. It fails to provide a characteristic value or a representative item where the distribution of the series is not normal & gives a 'U' shaped curve rather than a bell shaped one.

Geometric Mean

Geometric mean is a type of mathematical average. It is defined as the nth root of the product of the 'n' items. The fundamental formula for its computation stands as under:

$$G = \sqrt[n]{x_1 . x_2 . x_3 \cdots x_n}$$

Where G = geometric mean, n = no. of items and

$x_1\ x_2 \ldots x_n$ = values of the 1st, 2nd and so on items.

The above formula holds good, if the values of items are small, simple and capable of being factorised easily. But when the values of the variable are big or complicated ones, the geometric mean will have to be calculated by the logarithmic operation through following formula:

$$G = \text{Antilog of } \frac{1}{N}\{\log X_1 + \log X_2 + \log X_3 + \cdots + \log X_n\}$$

$$\text{Or, } G = A.L \frac{\sum \log X}{N}$$

Example: Find the geometric mean of 1, 8 and 64.

Solution:

The geometric mean of 1, 8 and 64 would be $G = \sqrt[3]{1 \times 8 \times 64} = 1 \times 2 \times 4 = 8$.

Example: Find the geometric mean of the series 5, 10, 15, 20, 25.

Solution: Computation of the Geometric Mean by logarithmic operation

X	Log X	
5	.6990	
10	1.0000	
15	1.1761	
20	1.3010	
25	1.3979	
Total	5.5740	N=5

By the formula of geometric mean, we have

$$G = AL \frac{\sum \log X}{N} = AL \frac{5.5740}{5} = AL\ 1.1148 = 13.02 = 13 \text{ (approx.)}$$

Notes

Harmonic Mean

Harmonic mean is another mathematical average. It is defined as the "reciprocal of the arithmetic average of the reciprocals of the values of a variable". Thus, the fundamental formula for its calculation in the different series is as follows:

$$H = \frac{N}{\Sigma r_{(x)}}$$

where H = Harmonic mean

$\sum r_{(x)}$ = Sum of the reciprocals of the variables or mid values

& N = No. of observations or Σf

The formula for calculating harmonic mean in discrete series is given by:

$H = \frac{N}{\Sigma \frac{f}{x}}$ where, N = total frequency

The formula for calculating harmonic mean in continuous series is given by:

$H = \frac{N}{\Sigma \frac{f}{m}}$ where N = total frequency and m = mid-points

Example: Find the harmonic mean of 1, 2, 3, 4, 5.

Solution: Calculation of Harmonic Mean

X	1/x
1 2 3 4 5	1/1 = 1.000 1/2 = 0.500 1/3 = 0.333 1/4 = 0.250 1/5 = 0.200
N = 5	Σ(1/x) = 2.283

We have H = $\frac{N}{\Sigma \frac{1}{x}} = \frac{5}{2.283} = 2.19$

Example: From the following data calculate the harmonic mean.

X:	10	20	30	40	50
f:	12	20	45	6	10

Solution: Calculation of Harmonic Mean

X	F	f/x
10	12	12/10 = 1.20
20	20	20/20 = 1.00
30	45	45/30 = 1.50
40	6	6/40 = 0.15
50	10	10/50 = 0.20
	N =93	Σ f/x = 4.05

We have H = $\frac{N}{\Sigma \frac{f}{x}} = \frac{93}{4.05} = 22.96$

Notes

Example: From the following information, calculate harmonic mean.

Class Interval	10-12	12-14	14-16	16-18	18-20
Frequency	5	6	10	4	3

Solution: Calculation of Harmonic Mean

Class intervals	Frequency	Mid-points	f/m
10-12	5	11	5/11 =0.454
12-14	6	13	6/13 = 0.464
14-16	10	15	10/15 = 0.667
16-18	4	17	4/17 = 0.235
18-20	3	19	3/19 = 0.158
	N =8		Σ f/m = 1.976

We have, H = $\frac{N}{\Sigma f/m} = \frac{28}{1.976} = 14.17.$

Example: A boy covered a square field by running 20 km.hr on the first side, then covered the second side by increasing his speed to 25 km/hr and third side by 30 km/hr and the last side at a speed of 40 km/hr. find his average speed.

Solution:

Here, N = 4 as there are 4 equal sides of a square field.

Speed = x =20, 25, 30, 40

$$H = \frac{N}{\frac{1}{20}+\frac{1}{25}+\frac{1}{30}+\frac{1}{40}} = \frac{4}{\frac{30+24+20+15}{600}} = \frac{4\times 600}{89} = 26.97\,\text{km/hr}$$

B. MEDIAN

Median is an average of position or a positional average. This is called so, because its value is determined with reference to its position in the value column of a series.

Median may be defined as the value of the middle item of a series arranged in ascending or descending order.

The fundamental formula for determining the median is given by

$$M = \text{Value of } \left(\frac{N+1}{2}\right) \text{th item}$$

where M represents the median,

m = median item i.e., $\left(\frac{N+1}{2}\right)$ th item

and N = number of items arranged in an order.

In case of individual and discrete series, the formula for median is given by,

$$M = \text{Value of item i.e., } \left(\frac{N+1}{2}\right) \text{th item}$$

In case of continuous series, the following two types of formula are to be used.

1. To find out the median item

   M = value of the $\left(\frac{N}{2}\right)$ th item

Notes

2. Interpolation formula

$M = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$ (if the series is in ascending order)

$M = L_2 - \frac{L_2 - L_1}{f_1}(m - c)$ (if the series is in descending order

where M = median

L_1 = lower limit and L_2 = upper limit of the median class,

f_1= frequency of the median class,

m = median item i.e., N/2 th item and

and C = cumulative frequency of the class that precedes the median class

Example: Determine the value of median from the following series:

X:	35	12	40	18	6	14	61

Solution:

The median is given by

M = Value of $\left(\frac{N+1}{2}\right)$ th item of item of the series arranged in an order

where, N or the number of items = 8

Thus, M = value of $\left(\frac{8+1}{2}\right)$ th item

= value of 4.5th item

= $\frac{1}{2}$(value of the 4th item + value of the 5th item)

The values arranged in the ascending order are:

11, 13, 15, 35, 55, 62, 71, 75

The values of the 4th item and 5th item located in the ascending order are 35 and 55 respectively.

Thus, M =$\frac{1}{2}(35 + 55) = 45.$

Example: Determine the value of median from the following series:

Marks:	0-10	10-15	15-20	20-25	25-30
No. of students	7	5	8	38	42

Solution: Determination of the Median

Marks	**No. of students**	**CF**
0-10	7	7
10-15	5	12
15-20	8	20
20-25	38	58
25-30	42	100
Total	N = 100	

Median = Value of 'm'

= Value of $\left(\frac{N}{2}\right)$ th item [∴ it is a continuous series]

Notes

= Value of $\left(\frac{100}{2}\right)$ th or 50th item that falls within the CF 58.

The value of this item lies in the class (20-25). By putting the formula of interpolation, we have

$M = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$

Thus, $M = 20 + \frac{25-20}{38}(50 - 20)$

$= 20 + \frac{5}{38} \times 30$

$= 20 + \frac{150}{38} = 20 + 3.95$

= 23.95 or 24 approx.

where M = median to be found out

L_1 = lower limit of the median class = 20

L_2 = upper limit of the median class =25

f_1 = frequency of the median class = 38

m = median item i.e., 50 and

C = cumulative frequency of the class preceding the median class = 20

Merits

1. It is simple to understand.
2. It can be determined graphically along with the quartiles etc.
3. It can be determined easily in open end series without estimating the lowest or highest class limits.

It is considered suitable for computation of the mean deviation as the sum of the deviations taken from it is the minimum.

Demerits

1. It is not rigidly defined & as such its value cannot be computed but located.
2. It is not based on all the observations of the series.
3. It is not capable of further algebraic treatment like mean, geometric mean & harmonic mean.
4. It is very much affected by fluctuations in sampling.

C. MODE

Mode is an average of position. It is defined as the value around which the items are most heavily concentrated. In short, it is the value which occur maximum times or against which maximum frequencies cluster around for maximum times.

For a continuous series, we use the formula of interpolation to find the mode. The formula is given as under:

$$Z = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2}(L_2 - L_1)$$

or $$Z = L_2 - \frac{f_1 - f_2}{2f_1 - f_0 - f_2}(L_2 - L_1)$$

where, Z = Mode,

L_1 = lower limit of the modal class,

L_2 = upper limit of the modal class,

f_1 = frequency of the modal class,

$f_{0 =}$ frequency of the class preceding the modal class,

and nf_2 = frequency of the class succeeding the modal class.

Notes

Example: From the following data relating to the daily wages of 15 workers in a factory determine the value of the modal wage:

Wages in Rs.	14	16	16	14	22	13	15	24	12	23	14	20	17	21	22

Solution:Determination of the modal wage of 15 workers by the method of inspection

Wages in ascending order X	12	13	14	15	16	17	20	21	22	23	24
No. of occurrences F	1	1	3	1	2	1	1	1	2	1	1

On an inspection of the above table, it is observed that the wage 14 has occurred maximum times i.e., 3. Hence, the modal wage is Rs. 14.

Example: From the following data determine the value of the mode by the appropriate method.

Marks	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
No. of students	5	15	25	10	5	4	3	3

Solution: Determination of the modal marks

By the method of inspection, it comes out that the class (15-20) has the maximum frequency 25 which is much more than its next maximum frequency 15. Hence, (15-20) is the modal class.

By putting the formula of interpolation,

$$Z = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2}(L_2 - L_1), \text{ we get}$$

$$Z = 15 + \frac{25-15}{2\times25-15-10}(20-15)$$

$$= 15 + \frac{10}{25} \times 5 = 17$$

$$\therefore \text{ Mode} = 17$$

Merits

1. It gives the most representative value of a series.
2. It is not affected by the extreme values of a series.
3. It is considered a reliable average for studying skewness of a distribution.
4. It is very much useful in the field of business & commerce as it helps a businessman in taking a decision on the varieties of the goods he should procure in large quantities to enhance his sales.

Demerits

1. It is not rigidly defined & so in certain cases it may come out with different results.
2. It is not based on all the observations of a series but on the concentrations of frequencies of the items.
3. It is not capable of further algebraic treatment like A.M, G.M or H.M.
4. It cannot be easily determined graphically if two or more values of a series have the same highest frequency.

Empirical Relationship between Mean, Median and Mode

In a uni-modal and symmetrical distribution, the value of mean, median and mode are equal. For the asymmetrical distribution (positively skewed or negatively skewed), Karl Pearson has suggested a relationship between mean, median and mode which is termed as empirical relationship.

Mode = 3 Median – 2 Mean or $Z = 3\,Me - 2\overline{X}$

Notes

2.8 PARTITION VALUES

Like median, there are a host of other positional averages which are determined just in the similar manner as that of median. They are:

a. Quartiles
b. Deciles
c. Percentiles
d. Octiles
e. Septiles
f. Quintiles
g. Hexiles

Of the 7 positional averages, the first three are used in practice in various economic and social problems.

- ***Quartiles:*** It divides a series into 4 equal parts and as such there can be 3 quartiles viz., Q_1, Q_2 and Q_3.
- ***Deciles:*** It divides a series into 10 equal parts and as such there can be 9 deciles viz., D_1, $D_2, \ldots\ldots, D_9$.
- ***Percentiles:*** It divides a series into 100 equal parts and as such there can be 99 percentiles viz., P_1, P_2, ..., P_{99}.

Calculation of Quartiles, Deciles and Percentiles

1. ***Individual observation series:*** First arrange the given data either in increasing order or decreasing order and then apply the following formula:

   For Quartiles: $Q_1 = \left(\frac{N+1}{4}\right)$ item, $Q_3 = 3\left(\frac{N+1}{4}\right)$th item

   For Deciles : $D_1 = \left(\frac{N+1}{10}\right)$th item, $D_6 = 6\left(\frac{N+1}{10}\right)$th item and so on.

   For Percentile: $P_1 = \left(\frac{N+1}{100}\right)$th item, $P_{10} = 10\left(\frac{N+1}{100}\right)$th item, $P_{90} = 90\left(\frac{N+1}{100}\right)$th item

   where N =No. of items

Example: Calculate Q_1, Q_3, D_6, P_{70} from the following data.

13, 16, 19, 15, 2, 8, 10, 9, 7, 18, 20

Solution:

Arranging the data in increasing order we get

2, 7, 8, 9, 10, 13, 15, 16, 18, 19, 20. Thus, N =11

So, $Q_1 = \left(\frac{N+1}{4}\right)$th item $= \frac{11+1}{4} = 3$rd item i.e., 8

$Q_3 = 3\left(\frac{N+1}{4}\right)$th item $= 3\left(\frac{11+1}{4}\right) = 9$th item i.e.,18

$D_6 = 6\left(\frac{N+1}{10}\right)$th item $= 6\left(\frac{11+1}{10}\right)$th item

$= 7^{th}$ item + 0.2 (8^{th} item -7^{th} item) =15 + 0.2 (16 -15) =15.2

$P_{70} = 70\left(\frac{N+1}{100}\right)$th item $= 70\left(\frac{11+1}{100}\right)$th item $= 70 + \left(\frac{12}{100}\right) = 8.4$th item

= 8th item + 0.4 (9^{th}-8^{th}) item = 16 + 0.4 (18 - 16) = 16.8

2. ***Discrete series:*** The formula remains the same but here N=total frequency. Here the cumulative frequencies are calculated and looking at the cumulative frequency corresponding to the quartiles, deciles and percentiles values obtained, the value of the variable corresponding to the cumulative frequency is the value of the quartile, decile or percentile.

Notes

3. ***Continuous series:*** In case of a continuous series, the value of required item will be determined through the relevant formula of interpolation.

Formula of Interpolation

For a Quartile $Q = L_1 + \frac{L_2 - L_1}{f_1}(q - C)$

For a Decile $D = L_1 + \frac{L_2 - L_1}{f_1}(d - C)$

For a Percentile $L = L_1 + \frac{L_2 - L_1}{f_1}(p - C)$

where, Q = the required quartile viz., Q_1, Q_2 or Q_3

q = the relevant quartile viz., q_1, q_2 or q_3

D = the required decile viz., D_1, D_2 or D_{3etc}

d = the relevant quartile viz., d_1, d_2 or d_9

P = the required percentile viz., P_1, P_2 or P_3

p = the relevant percentile viz., p_1, p_2 or p_{99}

f_1= the frequency of the relevant class

C = cumulative frequency of the relevant class

L_1 and L_2 = lower and upper limit of the relevant class interval

Example: Calculate quartiles, third deciles and 20^{th} percentile from the following data.

Wages	10-20	20-30	30-40	40-50	50-60	60-70
No. of persons	7	18	25	30	20	10

Solution: Calculation of Quartiles, D_3 and P_{20}

Class interval	f	c.f
10-20	7	7
20-30	18	25
30-40	25	50
40-50	30	80
50-60	20	100
60-70	10	110
	N=110	

Lower Quartile (Q_1)

$Q_1 = \left(\frac{N}{4}\right)$th item $= \frac{110}{4} = 27.5$th item

So Q_1 lies in the group (30-40).

$Q_1 = L_1 + \frac{L_2 - L_1}{f_1}(q - C)$

Here, $q = \frac{N}{4} = 27.5$, C =25, f_1 = 25, L_1 =30 and L_2 = 40

$\therefore\ Q_1 = 30 + \frac{40-30}{25}(27.5 - 25) = 30 + 1 = 31$

Upper Quartile (Q_3)

$Q_3 = \left(\frac{3N}{4}\right)$ th item $= \frac{3\times110}{4} = 82.5$th item

So, Q_3 lies in the group (50-60).

Notes

Here, q = 3N/4 = 82.5, C = 80, f_1 =20, L_1=50 and L_2 = 60

$\therefore Q_3 = L_1 + \frac{L_2-L_1}{f_1}(q-C) = 50 + \frac{60-50}{20}(82.5-80) = 50 + 12.5 = 51.25$

Third Decile D_3

$D_3 = 3\left(\frac{N}{10}\right)$ th item $= \frac{3\times110}{10} = 33$rd item

So, D_3 lies in the group (30-40).

Here, d = $\frac{3N}{10} = 33, C = 25, f_1 = 25, L_1 = 30$ and $L_2 = 40$

$\therefore D_3 = L_1 + \frac{L_2-L_1}{f_1}(d-C) = 30 + \frac{40-30}{25}(33-25) = 30 + 3.2 = 33.2$

20th Percentile P_{20}

$P_{20} = 20\left(\frac{N}{100}\right)$ th item $= \frac{20\times110}{100} = 22$nd item

So, P_{20} lies in the group (20-30).

Here, p = $\left(\frac{20\,N}{100}\right) = 22, C = 7, f_1 = 18, L_1 = 20$ and $L_2 = 30$

$\therefore P_{20} = L_1 + \frac{L_2-L_1}{f_1}(p-C) = 20 + \frac{30-20}{18}(22-7) = 20 + \frac{150}{18} = 20 + 8.33 = 28.33$

2.9 MEASURES OF DISPERSION

The word dispersion, literally, means deviation, difference or spread over of certain values from their central value. In relation to a statistical series, it refers to deviation of the various items of a series from their central value or the difference between any two extreme values of the series. Further, the word 'measure' means a method of measuring or ascertaining central values. Thus, the phrase 'measure of dispersion' means the various possible methods of measuring the dispersions or deviations of the different values from a designated value of a series which may be an average value or any other extreme value.

According to A.L. Bowley, "dispersion is the measure of variation of the items".

According to Simpson & Kafka, "the measure of scatterness of a mass of figures in a series about an average is called measure of variation or dispersion".

Generally, deviation of individual items from the central value is calculated then it is averaged to find out the degree of variation. Dispersion is termed as the average of second order, the central value being average of the first order.

Properties

1. It should be rigidly defined & free from any ambiguity.
2. It should be simple to follow & free from any jargon.
3. It should be easy for computation & free from complicated procedure of calculations.
4. It should be based on all the items of a series without ignoring any of them.
5. It should be capable of further algebraic treatment & should not violate any algebraic principle.
6. It should not be greatly affected by the values of the extreme items of a series.
7. It should not be affected by the fluctuation of sampling & should remain stable in all the cases of samples.

Notes

Merits

1. They indicate the dispersal character of a statistical series.
2. They speak of the reliability or dependability of the average value of a series.
3. They enable the statisticians for making comparison between two or more statistical series with regard to the character of their stability or consistency.
4. They enable one in controlling the variability of a phenomenon under his purview.
5. They supplement the measures of central tendency in finding out more & more information relating to the nature of a series.
6. They facilitate in making further statistical analysis of the series through the devices like coefficient of skewness, co-efficient of correlation, variance analysis etc.

Demerits

1. They are liable to misinterpretations & wrong generalisations by a statistician of biased character.
2. They are liable to yield inappropriate results as there are different methods of calculating the dispersion.
3. Excepting one or two, most of the methods of dispersion involve complicated process of computation.
4. They by themselves cannot give any idea about the symmetricity or skewed character of a series.

Methods of Studying Dispersion

1. Range
2. Inter-quartile Range and Quartile deviation
3. Mean deviation or Average deviation
4. Standard deviation
5. Lorenz curve method (Graphic Method)

The first two are called distance measures or positional measures. The third and fourth are termed as computed measures.

1. Range

Range is defined as the difference between the two extreme values of a series.

Thus, absolute range = largest value – smallest value

Or, R= L-S

And co-efficient of range or

Coeff . of Range = $\frac{L-S}{L+S}$

where, L = Largest value of a series.

S = Smallest value of a series

Range tells us the interval within which all items of a series lie. The average of two sets of data being the same, the distribution with less range has less dispersion.

Frequencies have no role in determining the value of range. So ignore them both in discrete as well as in continuous series.

Notes

Example: From the following data calculate the value of range and its coefficient.

Marks	10	25	30	35	45	50	60
No. of students	3	5	18	17	12	6	4

Solution:

Here, L= 60, S = 10. So, Range = L - S = 60 – 10 =50.

Coefficient of Range $= \frac{L-S}{L+S} = \frac{60-10}{60+10} = \frac{50}{70} = 0.71$

2. Inter Quartile Range

Inter-quartile range is defined as the difference between the two extreme quartiles of a series. Thus, absolute IQR = $Q_3 – Q_1$

And Coeff. of I Q R = $\frac{Q_3-Q_1}{Q_3+Q_1}$

Where Q_3 = Upper quartile of a series

Q_1 = Lower quartile of a series.

Semi-Inter Quartile Range or Quartile Deviation: Semi-inter quartile range or quartile deviation is defined as the average of the difference between the two extreme quartiles of a series. It is popularly called quartile deviation.

Thus, absolute quartile deviation is given by

$$Q.D = \frac{Q_3-Q_1}{2}$$

3. Mean Deviation

Mean deviation may be defined as the arithmetic average of the deviations of items of a series taken from its central value ignoring the plus & minus signs. The central value may be mean, median or mode. The signs are disregarded as sum of the deviations from arithmetic mean is always zero. It is also known as first moment of dispersion.

The mean deviation can be calculated in both absolute and relative manners. The fundamental formula for its computation stands as under.

M.D (δ) = $\frac{\Sigma|X-\bar{X}|}{N} = \frac{\Sigma|D|}{N}$

Where |D| (read as mod. D) = Modulos or absolute value of Deviation from Mean or Median or Mode ignoring signs, M.D = Mean deviation and N = No. of items or observations of a series.

Merits and Demerits

Merits:

1. It is simple to understand and easy to calculate.
2. Since it is based on all items of a series, it is a better method of measuring dispersion than range and quartile deviation.
3. It is not much very affected by the values of extreme items of a series.
4. It shows the dispersion of values around the measure of central tendency. It means it indicates how far each observation lies either from mean or median or mode.
5. Since deviations are taken from a central value, comparison about formation of different series can be easily made.
6. Average deviation from mean is always zero but mean deviation removes this problem by taking absolute values.

Notes

7. Because of its simplicity, it is extensively used in the field of commerce and economics to study the distribution of income and wealth.

Demerits

1. It is not rigidly defined in the sense that it is computed from any central value viz. mean, median mode etc. and thereby it can produce different results.
2. It violates the algebraic principle by ignoring the + and – signs while calculating the deviations of the different items from the central value of a series.
3. As signs are ignored, it is not suitable for further algebraic treatment.
4. It is affected much by the fluctuations in sampling.
5. It is rarely used in social sciences and for drawing statistical inferences.

Fields where the Mean Deviation is specially useful:

Mean deviation is specially useful and preferred in the following fields:

(a) In the matter of sample studies, preferably small samples, where detailed analysis is not necessary.

(b) In the preparation, analysis and interpretation of survey and reports and accounts meant for publication and presentation before the public most of whom are not expected to be well versed with the statistical methods.

(c) In the matter of forecasting the business cycles and economic conditions where much importance is given to the practical rather than theoretical events.

Example: The mark of 9 students in an examination out of 50 are

30, 20, 25, 27, 40, 48, 35, 31, 32.

Find mean deviation and its coefficient

(a) From mean.

(b) From median and interpret the result.

Solution:

(a) Calculation of Mean Deviation from Mean

X	$\|D\| = \|X - \bar{X}\|$
30	2
20	12
25	7
27	5
40	8
48	16
35	3
31	1
32	0
ΣX = 228	$\Sigma\|D\| = 54$

$\bar{X} = \frac{\Sigma X}{N} = \frac{228}{9} = 32$ M.D(from mean) $= \frac{\Sigma|D|}{N} = \frac{54}{9} = 6$

The mean deviation is 6 marks per student. It means the marks of the students deviate oa an average of 6 marks from the mean mark of 32.

Coefficient of M.D = $\frac{M.D}{\bar{X}} = \frac{6}{32} = 0.1875$

(b) Calculation of Mean deviation from Median

Notes

Arranging the series in increasing order, we get

20, 25, 27, 30, 31, 32, 35,40, 48. Here, N = 9

Me = $\left(\frac{N+1}{2}\right)$ th item $=\left(\frac{9+1}{2}\right)$ th item = 5th item = 31

Calculation of M.D from Median

X	$\lvert D\rvert = (X - Me)$
20	11
25	6
27	4
30	1
31	0
32	1
35	4
40	9
48	17
N = 9	$\Sigma\lvert D\rvert = 53$

M.D (from median) $=\frac{\Sigma|D|}{N}=\frac{53}{9}=5.88$

4. Standard Deviation

The concept of standard deviation was introduced by Karl Pearson in 1823. Standard deviation may be defined as the positive square root of the arithmetic average of the squares of deviations of the given values taken from the arithmetic average of a series. It is denoted by the Greek letter small sigma (σ).

$$\sigma = \frac{\sqrt{\sum(X-\overline{X})^{2}}}{N}$$

It is the best and most powerful measures of dispersion because:

(a) Squaring the deviations removes the drawback of ignoring ± sign (as done in mean deviation).

(b) It is also suitable for further mathematical treatment.

(c) Of all the measures, standard deviation is least affected by fluctuations in sampling.

Standard deviation is known as the Root-mean Square Deviation because it is the square root of the mean of the squared deviations from the actual mean. A small deviation means a high degree of uniformity and consistency in the observations of a series.

Coefficient of standard deviation is expressed as the ratio of the absolute standard deviation to the arithmetic average of the series.

Coeff. of standard deviation = $\frac{\text{Standard Deviation}}{\text{Mean}}$

Or, coeff. $\sigma = \frac{\sigma}{\overline{X}}$

It is a relative measure which is very useful in comparing the consistency of two or more series. If the value of the coefficient is less, a series is more consistent and vice-versa.

Notes

Example: Methods of Computation: Some of the important methods of computation are as under:

Method	**Simple series**	**Discrete and Continuous series**
1. Direct method (based on values)	$\sigma = \sqrt{\frac{\sum X^2}{N} - \left(\frac{\sum X}{N}\right)^2}$	$\sigma = \sqrt{\frac{\sum FX^2}{N} - \left(\frac{\sum FX}{N}\right)^2}$
2. Direct method (based on deviations from actual mean)	$\sigma = \sqrt{\frac{\sum x^2}{N}}$	$\sigma = \sqrt{\frac{\sum Fx^2}{N}}$
3. Short-cut method	$\sigma = \sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2}$	$\sigma = \sqrt{\frac{\sum Fd^2}{N} - \left(\frac{\sum Fd}{N}\right)^2}$
4. Step deviation method	$\sigma = \sqrt{\frac{\sum d'^2}{N} - \left(\frac{\sum d'}{N}\right)^2} \times c$	$\sigma = \sqrt{\frac{\sum Fd'^2}{N} - \left(\frac{\sum Fd'}{N}\right)^2} \times c$

Where σ = standard deviation of the items

Σx^2 = total of the squares of deviations of the items taken from their actual mean and $x=(X-\overline{X})$

N = total number of observations

ΣFx^2 = total of the products of the squares of deviations and their respective frequencies

ΣF = total of frequencies or N

Σd^2 =sum of the squares of deviations taken from the assumed mean i.e., $\Sigma(X - A)^2$

Σd = sum of the deviations taken from the assumed mean (X – A)

ΣFd^2 = sum of the products of the squares of deviation from the assumed mean and their corresponding frequencies

ΣFd = sum of the products of the deviations from the assumed mean and their corresponding frequencies

c = the common factor by which each of the deviations is divided to reduce its magnitude and $d' = \frac{X-A}{c} \text{ or } \frac{d}{c}$

Example: From the following simple series, find out the standard deviation.

X:	5	10	15	20	25	30	35

Solution: Computation of the S.D by the shortcut method

X	(X – A) = d	$(d)^2$	
5	-10	100	
10	-5	25	
15	0	0	
20	5	25	
25	10	100	
30	15	225	
35	20	400	
Total	35	875	N =7

Under the short cut method, the S.D is given by

$$\sigma = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2} = \sqrt{\frac{875}{7} - \left(\frac{35}{7}\right)^2} = \sqrt{125 - 25} = \sqrt{100} = 10$$

Coefficient of Variation

When the coefficient of standard deviation is expressed in percentage, it is termed as coefficient of variation.

Coefficient of Variation or C.V = $\frac{\sigma}{\bar{x}} \times 100$

A high coefficient of variation implies that the series is less consistent and a low coefficient of variation indicates more consistency or less variability.

Variance

Variance is the square of the standard deviation or V = σ^{2}.

The standard deviation, like any other device has certain merits & demerits. These are as follows:-

Merits

I. It is rigidly defined & free from any ambiguity.

II. It strictly follows the algebraic principle and never ignores the + & - sign like the mean deviation.

III. It is capable for further algebraic treatment as it has a lot of algebraic properties.

IV. It is used as a formidable instrument in making higher statistical analysis viz. co-relation, skewness, regression & sample studies etc.

V. It is not much affected by the fluctuations in sampling for which it is widely used in testing the hypothesis & for conducting the different tests of significance.

Demerits

I. It is not understood by a common man.

II. Its calculation is difficult as it involves many mathematical methods & processes.

III. It is affected very much by the extreme value of a series in as much as the squares of deviations of big items become proportionately bigger than the square of the small items.

5. Lorenz Curve

Lorenz curve is graphical method of studying the degree of dispersion of a series. This method was formulated by Dr. Max O. Lorenz, an economic statistician, for the measurement of economic inequalities such as in the distribution of wealth & income between different countries or between different periods of time. But now it is used to study the variation of profits, wages, sales, production or population, etc. It is a cumulative percentage curve in which the percentage of items is combined with the percentage of other variable such as profit, sales etc.

In drawing these curves, we use cumulative values of the values of the variables and the cumulative frequencies rather than their absolute values and frequencies. In a very simple manner, we draw a table of cumulative values of the data or observations and also their related cumulative frequencies. These cumulative are then converted into percentages of the totals and both these variables (cumulative percentage values) are then plotted as x and y values of graph. The line joining (0, 0) and (100, 100) is called the line of equal distribution and is used for comparison of the distribution variation of the observed data. The greater the departure of the curve from the line of equal distribution, the greater is the dispersion. Two or more series can be easily compared through Lorenz curves.

2.10. SUMMARY

- ***Quantitative Technique:*** Quantitative technique is a scientific approach to managerial decision making. This approach starts with data. Like raw material for a factory this data is manipulated or processed into information that is valuable to people for making decision.

Notes

- ***Statistics:*** Statistics are numerical facts in any department of inquiry placed in relation to each other.
- ***Average:*** An average is an attempt to find one single figure to describe the whole of figures.
- ***Data:*** Data are collection of any number of observations pertaining to a happening of either one or more variables.
- ***Frequency curve:*** A frequency polygon modification by smoothing classes and data points for a data set.
- ***Frequency Distribution:*** A table structuring the data into classes of suitable intervals showing number of observations falling into a certain class interval.
- ***Median:*** Median is a positional average because its value depends upon the position of an item and on its magnitude. It is that value of the variable which divides it into two equal parts.
- ***Mode:*** Mode is that value of the variable which occurs maximum number of times in a distribution and around which other items are densely distributed.
- ***Population:*** A collection of all the elements of the system under study.
- ***Sample:*** A collection of data of some elements of a population representing all its elements in right proportion.
- ***Representative sample:*** A sample containing all the relevant characteristics of the population it represents, in all its proportions the same as in original population.
- ***Binomial distribution:*** A distribution of data in which two values occur more frequently than the rest.
- ***Inter quartile range:*** The difference between the values of third and first quartiles.
- ***Measure of dispersion:*** the degree of variation of individual items from their central value is termed as dispersion. The central value may be mean, median or mode.
- ***Mean deviation:*** Arithmetic average of the absolute deviations ignoring ± sign of individual item from the central value is termed as mean deviation. It is also known as first moment of dispersion.
- ***Standard deviation:*** Positive square root of the variance or the square root of the mean of square deviations of the given observations from the arithmetic mean.
- ***Variance:*** A measure of mean squared variations or distance of the observations from their arithmetic mean.

2.11. SELF ASSESSMENT QUESTION

1. What is a quantitative technique?
2. Discuss classifications of quantitative technique?
3. Discuss the role and scope of quantitative methods for scientific decision in business management?
4. What is classification and what is its objective?
5. Distinguish between primary and secondary data?
6. What are the various methods of collecting statistical data?
7. The average monthly wage of all workers in a factory is Rs.444. If the average wages paid to male and female workers are Rs.480 and Rs.360 respectively find the percentage of male and female workers employed by the factory.
8. Find the average deviation about the median for the following distribution

   Demand: 5 15 20 25 30 35 40 45 50

Frequency: 3 2 6 8 10 5 8 7 5

9. From the data collected during industrial survey, as given in the table below, calculate the variance for the distribution.

Sales level:	10	15	20	25	30
Frequency:	10	5	15	4	6

10. Calculate median and mean deviation for the following frequency distribution.

Age:	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	40-45
No of person:	7	10	16	32	24	18	10	5	1

11. Calculate the coefficient of mean deviation from the following data

Height in inches:	50-53	53-56	56-59	59-62	62-65	65-68
No of persons:	2	7	24	27	13	3

12. Calculate the standard deviation from the following data.

Marks in MA:	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No of students:	5	14	7	12	9	6	2

13. The mean weight of a student in a group of students is 119 lbs. the individual weight of 5 of them are 115,109, 129, 117& 114 lbs. What is the weight of the 6th student?

14. Calculate the average daily wages for the workers of the two factories.

	Factory A	Factory B
No of wage earners:	350	300
Average daily wages (Rs.):	2	3

Notes

UNIT 3 PROBABILITY, CORRELATION AND REGRESSION

Structure

3.0 OBJECTIVES

After going through this unit you will be able to understand:

- Meaning and definition of probability
- Terminology used in probability application
- Types of probability and the laws
- Meaning and definition of correlation
- Methods and its coefficient
- Concept of probable error and coefficient of determination
- Linear regression
- Methods of calculation of regression coefficient
- How to use various formulas for solving practical problems

3.1 PROBABILITY CONCEPT

The term probability refers to 'an event' the happening and non-happening of which is uncertain or contingent. Literally, it means a chance a possibility a likelihood or an odd.

In usage, it is expressed in a statement as follows:

1. Possibly it will rain today.
2. There is a chance of your getting a first class.
3. This year's profit is likely to exceed of profits of all the preceding years.

Mathematically, it is a number which is expressed either in the form of a fraction, a percentage or a decimal. When the happening of the event is predicted to be certain, the value of the probability is taken to be unity i.e. 1 and when its happening is predicted to be impossible, the value of the probability is taken as 0 (zero). Thus, the value of a probability ranges from 0 to 1 and it is never negative.

Hence, the term probability may be defined “as a quantitative value of a chance that an event will take place in the face of favourable and unfavourable ways both of which are equally likely.”

Symbolically, it may be represented as

$$p\ (E) = \frac{m}{m + n}$$

Where, p (E) = the probability of happening of an event E

m = the number of favourable ways in which an event E can take place.

and n = the number of unfavourable ways in which an event cannot take place.

Further, the probability of not happening of an event can be represented as

$$q\ (E) = \frac{n}{m + n}$$

where q(E) represents the probability of not happening of an event E.

Sum of the probability of happening and that of not- happening would be equal to unity i.e., 1.

Symbolically, p (E) + q (E) = 1

Example: An unbiased die is thrown. Find the probability of its falling with the number 4 up.

Solution:

A die has six sides numbered 1, 2, 3, 4, 5, and 6. The number of favourable ways of getting a 4 up is 1 and the number of unfavourable ways of not getting a 4 up is 5. The total number of ways in which the die may fall is 6.

Here, m = 1, n = 5, m + n = 6 and the probability of happening of the event, i.e., getting 4 upward is given by

$$p\ (4) = \frac{m}{m + n} = \frac{1}{1 + 5} = \frac{1}{6}$$

and the probability of not happening of the event, i.e., not getting 4 is given by

$$q\ (4) = \frac{n}{n + m} = \frac{5}{1 + 5} = \frac{1}{6}$$

or, q(4) = 1 – p(4) = 1 – (1/6) =5/6.

3.2 SAMPLE SPACE

It is a set of all possible results, or outcomes of an experiment. It is represented symbolically by S = ( ). Thus, if two coins are tossed the variable possible outcomes are two heads – HH, two tails – TT, first one head & second one tail – HT, first one tail & second one head – TH.

The set of all these possible outcomes constitutes a sample space which is represented as thus, S = {HH, TT, HT, TH}

Each one of the possible results of an experiment represented as an element of a sample space is called a sample point. In the example given above, HH, TT, HT and TH are the different sample points belonging to the sample space — S.

Experiment

It is an operation which produces some result or outcome. Tossing a coin, rolling a die, drawing a ball from a bag of different balls, observing the defected items produced by a machine, & recording the number of customers visiting a shop are the example of experiment.

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Event

The outcome or a combination of outcomes under a definite rule is called an event. It is the subset of a sample space.

Example: In case of throwing a die, the sample space is (1, 2, 3, 4, 5, 6)

1. If we want even number, it will be (2, 4, 6), which is a sub set of a sample space.
2. For getting a number greater than 3, it will be (4, 5, 6).
3. For getting any one of the faces it will be (1), (2), (3), (4), (5), (6).

Equally likely Cases

If all possible outcomes have equal chance of occurrence, then such events are said to be equally likely. None of them is expected to occur in preference to others.

For example: in throwing a dice all the six faces (1, 2, 3, 4, 5, 6) have equal chance of turning up.

Favourable Cases

The cases which ensure occurrence of an event are called favourable cases.

For example:

1. In a toss of two coins, the number of cases favourable to the event 'exactly one head' is 2 i.e. HT, TH.
2. In drawing of spade from a pack of cards, the number of favourable cases is 13.

Dependent Events

A subsequent event, the probability or occurrence of which is affected by the probability or occurrence of its preceding event or events is called a dependent event. For example, when the cards are drawn consecutively without replacement from a pack of 52 playing cards, the probability of drawing a heart in the 2^{nd} draw will be (12/51) while in the first draw it was (13/52).

Mutually Exclusive Events

Two or more events are taken as mutually exclusive, if the happening of one excludes or prevents the happening of another at the same time. Thus, with the tossing of a coin, either a head or a tail can come up but both cannot come up at the same time.

Types of Probabilities

There are different types of probability which are discussed briefly as under:

1. ***Prior Probability:*** It is also known as the classical or mathematical probability which is associated with the games of chance, viz, tossing a coin, rolling a die, drawing a ball etc. The value of such a probability can bde stated in advance by expectation without waiting for the experiments as long as each outcome is equally likely to occur.

   According to Laplace, "Probability is the ratio of the number of 'favourable' cases to the total number of equally likely cases, if

   $$P(E) = \frac{\text{Number of favourable cases}}{\text{Total number of equally likely cases}} = \frac{m}{n}$$

   Limitations (a) Calculation of probability becomes difficult when $n \to \infty$.

   (b) It cannot be applied where the exhaustive cases are not equally likely.

   (c) It neither uses the data of actual experimentation nor the past experience.

2. ***Empirical probability:*** It is aldso otherwise known as statistical probability. It is used in practical problems, viz., computation of mortality table for a life insurance company.

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The classical approach may not explain the actual results in certain cases when the experiment is carried out a large number of times. Thus, when, the result of empirical probability approximates the probability given by classical approach.

Then, $p(E) = \lim_{n \to \infty} \frac{m}{n}$

Theoretically, we can never get the probability of an event from the above formula. If the number of observations is very large we can get a close estimate of p (E).

$p(E) = \frac{m}{n}$ (where n is sufficiently large)

This concept emphasises that probability is a large term concept. When the number of observations is very large, the empirical probability of an event is taken as the relative frequency of its occurrence and that such probability is equal to the prior probability.

3. ***Objective Probability:*** A probability which is calculated on the basis of some his torical data, common experience, recorded evidence or rigorous analysis is called objective probability. Probabilities of various events in rolling of dice or flipping of coins are examples of objective probability as they are based on dependable evidences.

4. ***Subjective Probability:*** A probability which is calculated on the basis of personal experience or opinion is called a subjective probability. According to this concept, probability is assigned to an event by an individual on the basis of his belief or degree of confidence placed in the occurrence of the event. In others words, a person's belief and the evidence available decide the probability of happening or not happening of an event. So, different persons put different values of probabilities for the happening of the same event. This method is particularly useful when objective data are not available.

5. ***Axiomatic Approach Probability:*** The definition of probability gives some axioms on which probability calculations are based. These are:

   (a) Probability of happening of an event ranges from 0 to 1. When an event cannot happen, its probability is 0 while for a certain event probability is 1.

   (b) Probability of the sample space S is one i.e., P (S) = 1. Therefore, if the probability of happening of an event A is described by P(A), then the probability of not happening will be $P_{(\bar{A})} = 1 - P_{(A)}$.

   (c) If A and B are two mutually exclusive events, the probability of occurence of either A or B is given by

$$P(AUB) = P(A) + P(B)$$

6. ***Conditional probability:*** The probability of a dependent event is called a conditional probability. If there are two dependent events, say A and B the conditional probability of A given that B has happened is given by

$$P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)}$$

The conditional probability of B given that A has happened is given by $P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)}$

Example: From a pack of playing cards two cards are drawn at random one after the other without replacement. Find the probability that both of them are court cards.

Solution:

There are 12 court cards, viz., 4 kings, 4 queens and 4 knaves and 52 total cards in a pack of playing cards.

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The probability of drawing a court card in the first instance is given by

$$P(A) = \frac{12}{52} = \frac{3}{13}$$

The probability of drawing a court card in the second instance is given by

$$P\left(\frac{B}{A}\right) = \frac{11}{51}$$

∴ The probability that both the cards drawn are court cards is given by

$$P(AB) = P(A) \times P\left(\frac{B}{A}\right) = \frac{3}{13} \times \frac{11}{51} = \frac{11}{221}$$

From this also, the $P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{11 \times 13}{221 \times 3} = \frac{11}{51}$

7. ***Joint Probability:*** The product of a prior probability and a conditional probability is the joint probability of any two dependent or independent events.
8. ***Marginal Probability:*** This is also called as unconditional probability which refers to the probability of occurrence of an event without waiting for the occurrence of another event.

   The marginal probability of the events, say A & B are represented as follows:

   Marginal Probability of A = P (A) = $\frac{m}{m+n}$

   Marginal probability of B = P (B) = $\frac{n}{m+n}$

Inter-relationship between the Conditional, Joint and Marginal Probabilities

All the three, conditional, joint and marginal probabilities are inter-related with each other by the relation: $P(B / A) = \frac{P(AB)}{P(A)}$

where P (B / A) = conditional probability of B given that A has happened,

P (AB) = joint probability of A and B

P (A) = marginal probability of A

3.3 RULES OF PROBABILITY

There are two important rules for computation of probabilities which are known as the 'Theorem of Probability'. They are:

1. The Addition Theorem (Rule of summation)
2. The Multiplication Theorem (Rule of Multiplication)

Addition Theorem of Probability

(a) If there are two mutually exclusive events A and B, the probability of either event A or B is the sum of their individual probabilities.

Mathematically, P (A or B) = $P_{(A)} + P_{(B)}$

Or $P(A \cup B) = P_{(A)} + P_{(B)}$

In this case, $P(A \cap B) = \phi$

(b) When events are not mutually exclusive, there are some common elements. If two events A and B are not mutually exclusive, then the probability of either A or B or both occurring is

Notes

equal to the sum of their individual probabilities minus the probability of A and B occurring together. Mathematically,

$$P(A \text{ or } B) = P_{(A)} + P_{(B)} - P(A \text{ and } B)$$

$$\text{or, } P(A \cup B) = P_{(A)} + P_{(B)} - P(A \cap B)$$

For three events, A, B and C,

$$P(A \cup B \cup C) = P_{(A)} + P_{(B)} + P_{(C)} - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Example: One card is drawn at random from a pack of 52 cards. What is the probability that it is either an ace or a jack?

Solution:

Probability of drawing an ace = $\frac{4}{52}$

Probability of drawing a jack = $\frac{4}{52}$

Since the two events are mutually exclusive, the probability that the card drawn is either an ace or a jack = $\frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

Example: A bag contains 20 balls marked with numbers 1to 20. One ball is drawn at random. Find the probability that it will be a multiple of 2 or 5.

Solution:

The probability of the number being a multiple of 2 is

$$P(2, 4, 6, 8, 10, 12, 14, 16, 18, 20) = \frac{10}{20}$$

The probability of the number being a multiple of 5 is

$$P(5, 10, 15, 20) = \frac{4}{20}$$

Since the events are not mutually exclusive, the probability of being a multiple of both 2 and 5 is

$$P(10, 20) = \frac{2}{20}$$

∴ The probability of a number being a multiple of 2 or 5 is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{10}{20} + \frac{4}{20} - \frac{2}{20} = \frac{12}{20} = \frac{3}{5} = 0.6$$

Multiplication Theorem of Probability

According to this theorem, the probabilities of two or more related events are multiplied with each other to find out the net probability of their joint occurrence.

This rule of multiplication is applied to the problems of compound events where:

1. The related events are independent of each other and
2. The related events are not mutually exclusive.

If A and B are two independent events, the probability that both will occur is equal to the product of their individual probabilities. Thus, if A and B are independent, then

$$P(AB) \text{ or } P(A \cap B) = P(A) \times P(B)$$

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Similarly for three independent events, A, B and C

$$P(ABC) \text{ or } P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

For two dependent events A and B, the multiplication theorem is given by

$$P(A \cap B) = P(AB) = P(A) \times P(A/B) = P(B) \times P(A/B)$$

Similarly for three events A, B and C, we have

$$P(ABC) = P(A) \times P(B/A) \times P(C/AB)$$

Example: Six coins are tossed. What is the probability that all will show a head?

Solution:

Let P(A) denote the probability of showing a head.

$\therefore P(A) = \frac{1}{2}$ in a throw

These are independent events. So, P(A and B) = P(A) × P(B)

So, the probability of getting 6 heads will be

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64} \text{ or}$$

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64} \text{ or } {}^6C_6 = \left(\frac{1}{2}\right)^6 . \left(\frac{1}{2}\right)^0 = \frac{1}{64}$$

Example: A bag contains 10 white and 8 black balls. Two balls are drawn at random one after another without replacement. Find the probability that both the balls drawn are black.

Solution:

Probability of drawing a black ball in the first instance is

$$P(A) = \frac{8}{18} = \frac{4}{9}$$

The probability of drawing a second black ball (given that the first ball drawn is black) is

$P(B/A) = \frac{7}{17}$ [one each is reduced from the black ball and the total]

∴ The probability that both the balls will be black is

$$P(AB) = P(A) \times P(B/A) = \frac{4}{9} \times \frac{7}{17} = \frac{28}{153}$$

3.4 INDEPENDENT EVENTS

An event, the occurrence of which does not depend upon the occurrence of any other event is called an independent event. For example, the results of tossing a coin, rolling a die or drawing a ball each time after replacement of the ball drawn earlier. In such cases, the probability of the subsequent events is not affected by the occurrence of their preceding events.

3.5 BAYE’S RULE

Baye’s rule or probability is known in different names viz. posterior probability, revised probability & inverse probability. This has been introduced by Thomas Bayes, an English mathematician in his work known as Bayesian theory published in 1763. This theory consists of finding the probability of an event by taking into account of a given sample information. Thus, a sample of 3 defective items out of 100 (event A) might be used to estimate the probability that a machine is not working properly (event B).

This is called posterior probability because it is calculated after information is taken into account. This is called revised probability as it is determined by revising the prior probabilities in the light of the additional information gathered. Further, this is called inverse probability also, as it consists of finding the probability of a probability.

If an event A can only occur in conjunction with n mutually exclusive and exhaustive events B_1, B_2,……., B_n, and if A actually happens, the probability that it was preceeded by an event B_i (for A conditional probabilities of A given B_1, A given B_2, …., A given B_n are known) and if marginal probabilities $P(B_i)$ are also known, then the posterior probability of event B_i given that event A has occurred is given by

$$P\left(\frac{B_i}{A}\right) = \frac{P\left(\frac{A}{B_i}\right) . P(B_i)}{\sum P\left(\frac{A}{B_i}\right) . P(B_i)} = \frac{P(B_i) . P\left(\frac{A}{B_i}\right)}{P\left(\frac{A}{B_1}\right) . P(B_1) + P\left(\frac{A}{B_2}\right) . P(B_2) + \ldots ..}$$

3.6 RANDOM VARIABLE

A random variable is a numerically valued function defined on a sample space. A sample space for a random experiment (the experiment whose outcomes depend on chance) is a set of all possible outcomes of the experiment. Thus, random variable is a function which takes real values which are determined by the outcomes of the random experiment. A random variable may be discrete or continuous. It is discrete if the set of all possible values is finite or can be organized in the form of a sequence, whereas a continuous variable is the one capable of taking all values in the interval.

3.7 SIMPLE CORRELATION AND REGRESSION ANALYSIS

Correlation

Correlation, in statistics, refers to relationship between any two or more variables viz. height & weight, rainfall & yield, price & demand, income & expenditure, wages & price index, production & employment etc. Two variables are said to be correlated if with a change in the value of one variable there arises a change in the value of another variable.

Measuring the degree of relationship between these variables is done through correlation analysis. After establishing the relationship, we can estimate the value of one variable for a given value of the other variable and vice-versa.

According of Croxton & Cowden, "When the relationship is of a quantitative nature, the appropriate statistical tool for discovering & measuring the relationship & expressing it in a brief formula, is known as correlation".

Thus, correlation is the association between two or more variables and correlation analysis refers to using the statistical tool to measure the degree and direction of such relationship.

Types of Correlation

There are different types of correlation which may be noted between any two or more variables. These different types may be ramified into the following classes:

1. Simple, partial & multiple correlations.
2. Positive & negative correlation.
3. Perfect & imperfect correlation.
4. Linear & non-linear correlation.

The above types of correlation are explained here:

1. ***Simple, partial & multiple correlation:*** When the relationship between any two variables only is studied, it is a case of simple correlation. For example - study of relationship between saving & investment. When the relationship between any two out of three or more variables is

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studied ignoring the effect of the other related variables it is a case of partial correlation. For example - if out of the three related variables, say marks in Statistics, Accountancy and English, we study the correlation between the two variables viz., marks in statistics and accountancy ignoring the effect of the other variable i.e. marks in English, it will be a case of partial correlation. On the other hand, when the relationship between two or more variables is studied at a time it is a case of multiple correlation. For example, when the joint effect of fertiliser and rainfall is studied on agricultural production, it is a case of multiple correlation.

2. ***Positive & negative correlation:*** When the value of both the variables under study move in the same direction, i.e. with an increase in the value of variable, the value of the other variable increases & with a decrease in the value of one variable the value of the other variable decreases, it is a case of positive correlation. On the other hand, when both the variables under study move in the opposite direction i.e. an increase in the value of one is followed by a decrease in the value of the other & a decrease in the value of one is followed by an increase in the value of the other. It is a case of negative correlation. Use of fertiliser and production of agricultural crops are positively correlated while price and quantity demanded is negatively correlated. It is to be noted that the data of positive correlation when plotted on a graph paper will give an upward curve whereas the data of negative correlation plotted on a graph paper will give a downward curve.

3. ***Perfect & imperfect correlation:*** When the values of both the variables under study change at a constant ratio irrespective of the direction, it is a case of perfect correlation. On the other hand, when the values of the variables under study change at different ratios, it is a case of imperfect correlation. When correlations are measured mathematically, the value of perfect correlation will be either +1 or -1 & the value of imperfect correlation will be between ±1.

4. ***Linear & non-linear correlation:*** When the data relating to correlation plotted on a graph paper give rise to a straight line, it is a case of linear correlation. This is possible only when there is perfect relationship or constancy in the ratio of changes between the values of variables. On the other hand, when the data of two variables plotted on a graph paper give out a curve of any direction it is a case of curvy-linear or non-linear correlation. Like the linear correlation, non-linear correlation can be either positive or negative in nature & accordingly it may give either an upward or downward curve when plotted on a graph paper.

Methods of Studying Correlation

There are different methods of studying correlation between any two or more series. But for measuring the correlation between any two variables i.e. simple correlation, selection of the suitable method may be made out of the following:

1. Diagrammatic method.
2. Graphic method.
3. Karl Pearson's co-efficient of correlation.
4. Spearman's co-efficient method.
5. Concurrent deviation method.
6. Least square method.

1. ***Diagrammatic method:*** Under this method a scatter diagram is drawn on the basis of the corresponding values of any two variables. The values of one of the variables are represented by the X axis & those of the other variable by the Y axis through natural scales on which equal subdivision represents equal values. For each of the pairs of the values of the variables a dot is plotted on the graph paper. The dots so plotted on the graph paper give an indication of the direction of the diagram. If the diagram appears to be upward from the left bottom to the right top it indicates the sign of positive correlation. If the diagram appears to be downward from the left top to the right bottom, it indicates the sign of negative correlation.

2. ***Graphic method:*** Under this method, graphs are drawn for each of the variables under study. Such graphs can be drawn either on natural scale or on semi-logarithmic or ratio scale

depending upon the size of the magnitude of the data. If the size of the magnitudes of the data appears to be very big the semi-logarithmic scales are advantageously used.

3. ***Karl Pearson's method of co-efficient of correlation:*** The British biometrician Prof. Karl Pearson has devised several formulae of algebraic nature for measuring not only the nature of correlation but also the exact extent of the correlation in numerical forms. For this, he represents the co-efficient of correlation through the letter 'r' & asserts that the value of his 'r' must be in between ±1.

The formula of coefficient of correlation, under the direct method based on values, where the size of the given values of the variables X and Y are small or all the values of the variables can be reduced to small size by change of their scale or origin, stands as under:

$$r = \frac{N\sum XY - \sum X.\sum Y}{\sqrt{N\sum X^2 - (\sum X)^2}\,.\sqrt{N\sum Y^2 - (\sum Y)^2}}$$

where, r = Pearson's co-efficient of correlation

X = given or reduced values of the first variable

Y = given or reduced values of the second variable and

N = number of pairs of observations

The formula of correlation of coefficient, under the method based on deviations is measured on the basis of deviations of the items obtained from their respective actual arithmetic averages. Under this method, the coefficient of correlation is defined as the ratio of covariance between the two variables to the product of their standard deviation. The formula is given as under:

$$r = \frac{\sum xy}{N\sigma_x\sigma_y} = \frac{\sum xy}{\sqrt{\sum x^2 . \sum y^2}}$$

When deviation is taken from assumed mean, the formula for correlation coefficient is given as:

$$r = \frac{N\sum d_x d_y - (\sum d_x)(\sum d_y)}{\sqrt{N\sum {d_x}^2 - (\sum d_x)^2}\,.\sqrt{N\sum {d_y}^2 - (\sum d_y)^2}}$$

Properties of Karl Pearson's Coefficient of Correlation

Prof. Karl Pearson's coefficient of correlation has the following algebraic properties:

i. Its value must lie between + 1 & –1 i.e. $-1 \leq r \leq +1$. This property provides us with a yardstick of checking the accuracy of the calculations.

ii. It is independent of the changes of origin & scale as well.

iii. It is independent of the units of the measurement.

iv. It is independent of the order of comparison of the two variables. Symbolically $r_{xy} = r_{yx}$

v. It is the geometric mean of the two regression co-efficient i.e. $r = \sqrt{b_{xy} \times b_{yx}}$

Example: Calculate Karl Pearson's coefficient of correlation from the following data.

X:	2	3	4	5	6
Y:	9	6	5	3	2

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Solution: Calculation of Karl Pearson's coefficient of correlation

X	X^2	Y	Y^2	XY
2	4	9	81	18
3	9	6	36	18
4	16	5	25	20
5	25	3	9	15
6	36	2	4	12
ΣX = 20	$\Sigma X^2 = 90$	ΣY =25	$\Sigma Y^2 = 155$	ΣXY =83

$$r = \frac{N\sum XY - \sum X.\sum Y}{\sqrt{N\sum X^2 - (\sum X)^2}.\sqrt{N\sum Y^2 - (\sum Y)^2}}$$

$$= \frac{5(83)-(20)(25)}{\sqrt{5(90)-(20)^2}.\sqrt{5(155)-(25)^2}} = \frac{415-500}{\sqrt{450-400}.\sqrt{775-625}} = \frac{-85}{\sqrt{50}.\sqrt{150}} = \frac{-85}{86.6} = -0.98$$

There are different methods of calculating Karl Pearson's coefficient of correlation.

(a) **Actual Mean Method:** This method is followed when actual mean of both X and Y are not fractions. Pearson has devised the following formula to calculate the coefficient of correlation.

$$r = \frac{\sum xy}{\sqrt{\sum x^2.\sum y^2}} = \frac{\sum xy}{N\sqrt{\frac{\sum x^2}{N}}.\sqrt{\frac{\sum y^2}{N}}} = \frac{\sum xy}{N\sigma_x.\sigma_y} = \frac{cov.(X,Y)}{\sigma_x\sigma_y}$$

where $x = X - \overline{X}$, $y = Y - \overline{Y}$, σ_x = SD of X and σ_y = SD of Y.

This formula is also known as product moment method because it is based on the product of first moment about mean of the two variables.

(b) **Assumed Mean Method:** When actual mean of X and Y or both are fractions, the above method of calculating correlation becomes difficult and time consuming. To do away with this problem, we use the assumed mean method by applying the following formula.

$$r = \frac{N\sum d_x d_y - (\sum d_x)(\sum d_y)}{\sqrt{N\sum d_x{}^2 - (\sum d_x)^2}.\sqrt{N\sum d_y{}^2 - (\sum d_y)^2}}$$

Where, $d_x = X - A_x$, $d_y = Y - A_y$, N = No. of paired observations and A_x and A_y are the assumed mean of X and Y.

4. ***Spearman's rank correlation:*** This method is a development over Karl Pearson's method of correlation on the point that,

- It does not need the quantitative expression of the data.
- It does not assume that the population under study is normally distributed.

Under this method, correlation is measured on the basis of the ranks rather than the original values of the variables. For this, the values of the two variables are first converted into ranks in a particular order.

Spearman devised the following formula:

(a) When no two individuals are awarded the same rank.

$$r = 1 - \frac{6\sum d^2}{n^3 - n} = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

where d = difference between the pair of ranks of the same individual ($R_1 - R_2$)

n = number of pairs and

r = coefficient of correlation.

Notes

(b) When two individuals are given the same rank: When there is more than one item with the value rank in either or both the series, an average rank is given to all the equal items. In case of equal ranks an adjustment is made to the above formula by adding $\frac{1}{12}(m^3 - m)$ to the value of $\sum d^2$ as;

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1{}^3 - m_1) + \frac{1}{12}(m_2{}^3 - m_2) + \cdots\cdots\right]}{n(n^2-1)}$$

where m = No. of items whose ranks are equal.

Example: Ranking of 10 trainees at the beginning (*x*) and at the end (*y*) of a certain course are given below.

Trainees	A	B	C	D	E	F	G	H	I	J
X:	1	6	3	9	5	2	7	10	8	4
Y:	6	8	3	7	2	1	5	9	4	10

Calculate spearman's coefficient of correlation.

Solution:

Calculation of Rank Correlation Coefficient

Trainees	**X(R_1)**	**Y(R_2)**	**d = (R_1-R_2)**	**d^2**
A	1	6	-5	25
B	6	8	-2	4
C	3	3	0	0
D	9	7	2	4
E	5	2	3	9
F	2	1	1	1
G	7	5	2	4
H	10	9	1	1
I	8	4	4	16
J	4	10	-6	36
			Σd = 0	$\Sigma d^2 = 100$

We have,

$$r = \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(100)}{10^3 - 10} = 1 - \frac{660}{990} = 1 - 0.606 = 100.394$$

Example: From the following data calculate rank coefficient of correlation.

X:	50	35	42	11	18	18	67	26	18	59
Y:	15	15	26	8	17	6	22	11	8	21

Solution:

Computation of Rank Correlation Coefficient

X	**R_1**	**Y**	**R_2**	***d* = (R_1 - R_2)**	***d^2***
50	8	15	5.5	2.5	6.25
35	6	15	5.5	0.5	0.25
42	7	26	10	-3	9
11	1	8	2.5	-1.5	2.25
18	3	17	7	-4	16

Notes

18	3	6	1	2	4
67	10	22	9	1	1
26	5	11	4	1	1
18	3	8	2.5	0.5	0.25
59	9	21	8	1	1
n =10	``			$\Sigma d = 0$	$\Sigma d^2 = 41$

In both the series, lowest items are given rank 1.

In X – series, the item 18 is repeated thrice. So average rank = $\frac{2+3+4}{3} = 3$ is given. The next item 26 is given the 5th rank.

In Y – series, the item 8 is repeated twice. So, the average rank = $\frac{2+3}{2} = 2.5$ is given. the next item 11 is given the 4th rank. Again, the item 15 is repeated twice. So, the average rank $= \frac{5+6}{2} = 5.5$ is given. The next item 17 is given the 7th rank.

So, in this problem m takes the value of 3, 2 and 2 i.e., $m_1 = 3$, $m_2 = 2$, $m_3 = 2$.

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^2 - m_3)\right]}{n(n^2 - 1)}$$

$$= 1 - \frac{6\left[41 + \frac{1}{12}(3^2 - 3) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2)\right]}{10(100 - 1)}$$

$$= 1 - \frac{6\left[41 + \frac{24}{12} + \frac{6}{12} + \frac{6}{12}\right]}{990} = 1 - \frac{6(41 + 2 + 0.5 + 0.5)}{990} = 1 - \frac{6 \times 44}{990} = 1 - 0.267 = 0.733$$

5. Concurrent Deviation Method

This method is a development over the rank correlation method in the sense that its process of calculation is the simplest & shortest of all the algebraic methods discussed above. Any number of observations can be easily solved under this method.

$$r_{(C)} = \pm\sqrt{\pm\frac{2C - n}{n}}$$

where, r (c) = co-efficient of concurrent deviation

C = number of concurrent deviation

n = number of pairs of deviation.

6. Methods of Least Square

Under this method, the coefficient of correlation can be obtained by the following formula:-

$$r = \sqrt{b_{xy} \times b_{yx}}$$

where $b_{xy} = \frac{\sum xy}{\sum y^2}$ and $b_{yx} = \frac{\sum xy}{\sum x^2}$

Under this method, another formula for the coefficient of correlation is given by:

$$r = \frac{S_y}{\sigma_y}$$

Notes

where S_y = standard deviation of the computed Y series

σ_y = standard deviation of the original Y series

The computed Y series is found out on the basis of the linear equation Y = a + bX, where a and b are the constants, the values of which are determined by solving the following two simultaneous equations:

$$\Sigma Y = Na + b\,\Sigma X$$

$$\Sigma XY = a\,\Sigma X + b\,\Sigma X^2$$

Probable Error in Correction

The data given to find the coefficient of correlation are usually drawn from samples. Therefore, they like other statistical quantities are subject to errors of sampling. Thus error in sampling is termed as probable error. The probable error can be obtained by applying the following formula:

$$P.E = 0.6745\,\frac{1-r^2}{\sqrt{N}} = \frac{2}{3} \times \frac{1-r^2}{\sqrt{N}};$$

where r = coefficient of correlation,

N = no. of pairs of observations.

Significance

(a) If r > P.E, the existence of correlation is certain.

(b) It helps us to analyse whether the coefficient of correlation is significant or not.

(i) If r < 6 P.E, correlation is insignificant.

(ii) If $r \geq 6$ P.E, correlation is significant or if $\frac{r}{P.E} \geq 6$, the correlation is said to be significant.

(c) With the help of probable error, we can find the lower and upper limit within which the coefficient of correlation in the population can be expected.

(d) Limit of the correlation of the population = $r \pm$ P.E

Standard Error: It is defined by the following formula.

$$S.E = \frac{1-r^2}{\sqrt{N}}\ or\ SE = \frac{3}{2}P.E$$

Coefficient of Determination

Coefficient of determination is another measure of determining the correlation in the same manner as that of the Pearson's co-efficient of correlation. This has been introduced by the famous statistician Tuttle. Coefficient of determination is nothing but the square of the Pearson's coefficient of the correlation i.e. r^2. In other words it is defined as the ratio of the explained variations to the total variations. Coefficient of determination is represented as follows:

$$r^2 = \frac{\text{Explained Variation}}{\text{Total Variation}}$$

From the above formula it is to be noted that the value of the coefficient of determination shall always remain positive & its maximum value will be 1.

For example, if r = 0.9, the coefficient of determination will be $(0.9)^2 = 0.81$. it means 81% of the variance of the variable Y is caused by X. In other words, 19% of the variance in Y is due to other factors.

Co-efficient of Non-determination

Co-efficient of non-determination is derived from the co-efficient of determination. It is defined by Tuttle as one minus the co-efficient of determination. Symbolically it is represented by K^2 and $K^2 = 1 - r^2$ Where r is the coefficient of correlation.

Notes

Thus, coefficient of non-determination is the ratio of the unexplained variation to the total variation.

Coefficient of alienation

This is a development over the co-efficient of determination. This is symbolically represented by K as the square root of the co-efficient of non-determination.

Thus, the co-efficient of alienation or $k = \pm\sqrt{1 - r^2} = \pm\sqrt{1 - \frac{\text{Explained Variation}}{\text{Total variation}}}$

3.8 REGRESSION ANALYSIS

Regression analysis is a statistical technique that expresses the average relationship between two variables in the form of a mathematical equation. It establishes the average relationship between two or more variables in the form of an equation through which the value of one variable can be predicted for a given value of the other variable. The variable whose value is estimated is termed as dependent variable and the variable that is used to estimate the dependent variable is called independent variable. The independent variable is, sometimes referred to as 'regressor' or 'predictor' or 'explanatory variable' and the dependent variable is called the 'regressed' or 'explained variable'. For example, while estimating sales in relation to advertisement expenditure, we generally take advertisement expenditure as independent variable and sales as dependent variable.

The dictionary meaning of the term 'regression' is stepping back or returning to the average value. The term was first used by Sir Francis Galton — a British biometrician in course of his studying the relationship between the heights of fathers and sons.

Definition

In the words of M. M. Blair, "Regression is the measure of the average relationship between two or more variables in terms of the original units of data".

Characteristics of Regression Analysis

The essential characteristics of regression analysis are the following:

1. It consists of mathematical devices that are used to measure the average relationship between two or more closely related variables.
2. It is used for estimating the unknown values of some dependent variable with reference to the known values of its related independent variables.
3. It provides a mechanism for prediction or forecast of the values of one variable in terms of the values of the other variable.

It consists of two lines of equation viz. (i) equation of X on Y and (ii) equation of Y on X.

Types of Regression Analysis

There are different types of regression analysis which can be made between two or more related variables. They can be grouped into the following three classes of dichotomy:-

1. ***Simple & multiple regression analysis:*** A simple regression analysis is one which is defined to only two variables say, X & Y or price &demand, or advertisement expenditure & volume of sales etc. In such cases, the value of one variable is estimated on the basis of the value of another variable. The variable whose values are estimated is called dependent, regressed or explained variable & the variable that serves as the basis of determining the value of the other variable is called the independent, regressing or explanatory variable.

   A multiple regression analysis, on the other hand, is one which is made among more than two related variables at a time say, X, Y& Z or say over sales, price & income of the people. In such analysis, the value of one variable say, sales are estimated on the basis of the other remaining variables say price of goods & income of the consumer.

Notes

2. ***Linear & non-linear regression analysis:*** A linear regression analysis is one which gives rise to a straight line when the data relating to the two variables are plotted on a graph paper. This happens, when the two variables have linear relationship with each other which means that with a change in the value of the independent variable by one unit, there occurs a constant change in the values of the dependent variable.

   A non-linear regression analysis, on the other hand, is one which gives rise to a curved line when the data relating to two variables are plotted on a graph paper. In such cases, the regression equation will be a function involving the terms of higher order like, $Y=X^2$, $Y=X^3$ etc.

3. ***Total & partial regression analysis:*** A total regression analysis is one which is made to study the effect of all the important variables on one another. For example, when the effect of advertising expenditure, income of the people & price of the goods on the volume of sales are measured, it is a case of total regression analysis.

   A partial regression analysis on the other hand, is one which is made to study the effect of one or two relevant variables on another variable.

Methods of Studying Regression Analysis

There are two different methods of studying simple (linear and partial) regression between two related variables. They are:

1. Graphic method
2. Algebraic method

1. ***Graphic Method:*** Under this method, one or two regression lines are drawn on a graph paper to estimate the values of one variable say X on the basis of the given values of another variable say, Y. If there is a perfect correlation between the two variables, only one regression line can be drawn, for in that case both the regression lines of X on Y & Y on X will coincide. In case, the correlation between the two variables is not perfect, two lines of regression are to be drawn on the graph paper one of which will be the regression line of X on Y and the other will be the regression line of Y on X. If there is no correlation between the two variables, then the two lines of regression will be perpendicular to each other.

   The line of regression is a graphic line which gives the best estimate of one variable for any given value of the other variable. Such a line can be drawn on a graph paper by any one of the following two methods:

   (a) Scatter diagram method

   (b) Method of least squares.

   (a) *Scatter diagram method:* Under this method, one or two regression lines are drawn on a graph paper to estimate the values of one variable say X on the basis of the given values of another variable say, Y. If there is a perfect correlation between the two variables, only one regression line can be drawn, for in that case both the regression lines of X on Y & Y on X will coincide. In case, the correlation between the two variables is not perfect, two lines of regression are to be drawn on the graph paper one of which will be the regression line of X on Y and the other will be the regression line of Y on X. If there is no correlation between the two variables, then the two lines of regression will be perpendicular to each other.

   (b) *Method of Least squares:* This method is a development over the scatter diagram. Under this method we are to draw the lines of best fit as the lines of regression. These lines of regression are called the lines of the best fit because with reference to these lines we can get the best estimates of the values of one variable for the specified values of the other variable. Further this method is called as such because under this method the sum of the squares of the deviations between the given values of a variable & its estimated values given by the concerned line of regression is the least or minimum possible.

Notes

Under this method, the line of best fir for Y on X (the line of regression of Y on X) is obtained by finding the value of Y for any two (preferably the extreme ones) values of X through the following linear equations:

$$\mathbf{Y = a + b\,X,}$$

where a and b are the two constants whose values are to be determined by solving simultaneously the following two normal equations:

$$\mathbf{\Sigma Y = Na + b\,\Sigma X}$$

$$\mathbf{\Sigma XY = a\,\Sigma X + b\,\Sigma X^2}$$

where X and Y represent the given values of the X and Y variables respectively.

Similarly, the line of best fit of X on Y (the regression line of X on Y) is obtained by finding the values of X for any two (preferably the extreme ones) values of Y through the following linear equation:

$$\mathbf{X = a + b\,Y}$$

where the values of the two constants a and b are determined by solving simultaneously the following two normal equations:

$$\mathbf{\Sigma X = Na + b\,\Sigma Y}$$

$$\mathbf{\Sigma XY = a\,\Sigma Y + b\,\Sigma Y^2}$$

2. ***Algebraic Method:*** **U**nder this method, the two regression equations are formulated to represent the two regression lines or the lines of estimates respectively viz.,

   (a) The regression line of X on Y and

   (b) The regression line of Y on X

   To obtain such equations, we are to apply any of the following algebraic methods:

   (a) Normal equation method

   (b) Method of deviation from the actual means and

   (c) Method of deviation from the assumed means.

   (a) *Normal equation method:* this method is similar to that of the method of least square explained above except that the required values of a variable are estimated directly by the formulated equations rather than through the lines of estimates drawn on a graph paper.

   Under this method, the two regression equations viz.,:

   $$Y = a + b\,X \text{ ii) } X = a + b\,Y$$

   are obtained on the basis of the two normal equations cited under the method of least squares. Here, the regression equation $Y_e = a + b\,X$ is constructed to compute the estimated values of the Y variable for any given value of the X variable. In this formula,

   Y_e = estimated value of Y variable

   a = Y- intercept at which the regression line crosses the Y-axis i.e., the vertical axis. Its value remains constant.

   b = slope of the straight line and it represents a change in the Y variable for a unit change in the X variable. Its value remains constant for any given straight line

   and X = a given value of the X variable for which the value of Y is to be computed.

   The above equation $Y_e = a + b\,X$ is to be formulated on the basis of the following two normal equations:

   $$\Sigma Y = Na + b\,\Sigma X$$

   $$\Sigma XY = a\,\Sigma X + b\,\Sigma X^2$$

Notes

In this formula, ΣY = total of the given values of Y variable,

ΣX = total of the given values of X variable,

ΣX^2 = total of the squares of the given values of X variable,

ΣXY = total of the product of the given values of X and Y variable and

a and b = the two constants.

The above two normal equations are to be solved simultaneously to determine the values of the two constants a and b that appear as the essential factors in the regression equation.

Similarly, the other equation, $X_e = a + b\,Y$ is developed to compute the estimated values of the X variable for any given values of the Y variable.

(b) *Method of deviation from the actual means:* Under this method, the two regression equations are developed in a modified form from the deviation figures of the two variables from their respective actual means rather than their actual values. For this, the two regression equations are modified as under:

(i) Regression equation of X on Y: $X = \overline{X} + b_{xy}(Y - \overline{Y})$ or $X - \overline{X} = b_{xy}(Y - \overline{Y})$

(ii) Regression equation of Y on X: $Y = \overline{Y} + b_{yx}(X - \overline{X})$ or $Y - \overline{Y} = b_{yx}(X - \overline{X})$

In the above formula,

X = given value of the X variable

Y = given value of the Y variable

$\overline{X}$ = arithmetic average of the X variable

$\overline{Y}$ = arithmetic average of the Y variable

And b_{xy} = regression coefficient of X on Y i.e., $r\frac{\sigma_x}{\sigma_y}$

$$\therefore\ b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

where, r = correlation coefficient

σ_x = standard deviation of X variable =

σ_y = standard deviation of Y on X

and b_{yx} = regression coefficient of Y on X i.e., $r\frac{\sigma_y}{\sigma_x}$

$$\therefore\ b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

(c) *Method of Deviation from Assumed Mean:* This method is also known as short-cut method. Under this method, the regression between any two related variables is studied on the basis of the deviation of the items from their respective assumed means rather than their actual values or deviations from their respective actual means.

The formula for the two regression equations remains the same as cited above under the method of deviations from the actual means except that the two regression coefficients are determined by the following methods:

$$i)\ b_{xy} = \frac{N\sum d_x d_y - \sum d_x . \sum d_y}{N\sum {d_y}^2 - \left(\sum d_y\right)^2} \quad ii)\ b_{yx} = \frac{N\sum d_x d_y - \sum d_x . \sum d_y}{N\sum {d_x}^2 - (\sum d_x)^2}$$

Notes

In the above formula, N = number of pairs of observations,

d_x = deviations of items of X series from its assumed mean,

d_y = deviations of items of Y series from its assumed mean

Where $d_x = X - A_x$ and $d_y = Y - A_y$

Regression Coefficient

A regression coefficient is a vital factor that measures the change in the value of one variable with respect to a unit change in the value of another variable.

Regression coefficient of X on Y, $b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\sum xy}{\sum y^2} = \frac{cov.xy}{\sigma_y{}^2}$

Regression coefficient of Y on X, $b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\sum xy}{\sum y^2} = \frac{cov.xy}{\sigma_x{}^2}$

Properties of Regression Coefficients

1. The two regression coefficients must have the same algebraic sign i.e., both the regression coefficients will be either positive or negative.
2. The geometric mean of the two regression coefficients gives the coefficients of correlation i.e., $r = \sqrt{b_{xy} \times b_{yx}}$
3. The coefficient of correlation will have the same sign as that of regression coefficients.
4. If the value of regression coefficient is greater than 1, the value of the other regression coefficient will be less than 1.
5. The arithmetic mean of the two regression coefficients is more than or equal to the value of the coefficient of correlation i.e., $\frac{b_{xy} + b_{yx}}{2} \geq r$.
6. Regression coefficients are independent of the change of origin and are affected by the change of scale.

Advantages

1. ***Helps prediction:*** It helps in developing regression equations. Through this mathematical relationship, the value of a dependent variable can be predicted for a given value of independent variable.
2. ***Measures the Accuracy of estimation:*** standard error can be calculated through regression analysis which tells us the degree of the accuracy of the estimated figures. A low standard error implies a good estimate while a high standard error of estimate conveys otherwise.
3. ***Facilitates Calculation of r:*** The value of the coefficient of correlation can be calculated by using the two regression coefficients. Further, r^2 (the coefficient of determination) measures the degree of association of correlation that exists between two variables. A greater r^2 implies a good fit.

Limitations

1. It ignores the influence of other variables while studying the average relationship between two variables.
2. If the value of r is not significant, the regression equations are not good fit, hence should not be used for prediction.

Example: From the following data, calculate the two regression coefficients and write the two regression equations.

X Age of cars (in years)	2	4	6	8
Y Maintenance cost (Rs. ′000)	10	20	25	30

Notes

Solution: Calculation of Regression Coefficients

Age (X)	M.cost (Y)	X^2	Y^2	XY
2	10	4	100	20
4	20	16	400	80
6	25	36	625	150
8	30	64	900	240
$\Sigma X = 20$	$\Sigma Y = 85$	$\Sigma X^2 = 120$	$\Sigma Y^2 = 2025$	$\Sigma XY = 490$

$$\bar{X} = \frac{\sum X}{N} = \frac{20}{4} = 5, \bar{Y} = \frac{\sum Y}{N} = \frac{85}{4} = 21.25$$

Regression coefficient of Y on X,

$$b_{yx} = \frac{\sum XY - N.\bar{X}.\bar{Y}}{\sum X^2 - N(\bar{X})^2} = \frac{490 - 4(5)(21.25)}{120 - 4(5)^2} = \frac{490 - 425}{20} = \frac{65}{20} = 3.25$$

So, the regression equation of Y on X will be:

$Y - \bar{Y} = b_{yx}(X - \bar{X})$

$\Rightarrow Y - 21.25 = 3.25\ (X - 5) \Rightarrow Y - 21.25 = 3.25\ X - 16.25 \Rightarrow Y = 3.25\ X + 5$

Regression coefficient of X on Y,

$$b_{xy} = \frac{\sum XY - N.\bar{X}.\bar{Y}}{\sum Y^2 - N(\bar{Y})^2} = \frac{490 - 425}{2025 - 4(21.25)^2} = \frac{65}{218.75} = 0.30$$

So, the regression equation of X on Y will be:

$X - \bar{X} = b_{xy}(Y - \bar{Y})$

$\Rightarrow X - 5 = 0.30(Y - 21.25) \Rightarrow X - 5 = 0.30Y - 6.375 \Rightarrow X = 0.30Y - 1.375$

Ratio of Variation

By ratio of variation we mean the arithmetic average of the ratio of the percentage deviations from the mean in the relative series as compared to those in the subject. It gives us an idea about the variation in the relative series as compared to a constant variation in the subject series. It may be noted that the variable in which the average percentage deviation is less is generally taken as the relative series so that the value of the ratio of variation may be less than unity.

3.9 SUMMARY

The knowledge of the theoretical probability distribution is of great use in the understanding and analysis of a large number of business and economic situation.

For example with the use of use of probability distribution it is possible to test a hypothesis about a population to take decision in the face of uncertainty, to make forecast, etc.

Binomial distribution is a theoretical probability distribution which was given by James Bernoulli. These distributions are applicable to situations with the following characteristics.

1. An experiment consists of a finite number of repeated trials.
2. Each trial has only two possible outcomes whish are termed as a 'success' or a 'failure'

Notes

3. The probability of a success denoted by p which is remain constant trial to trial. The probability of failure is denoted by q is equal to 1-p.

Correlation is a statistical tool which studies the relationship of two variables. It involves various methods and techniques used for studying and measuring the extent of the relationship of the two variables.

The term regression was first used sir Francis Galton in connection with the studies on the estimation of the nature of the sons of tall parents to its effect on the mean population.

Now this expression is used in statistics for estimation or prediction of an unknown value of one variable from the known value of the other variable. This powerful tool is now used extensively in natural, social or physical sciences.

Random variable: a variable that takes on different values as a result of the outcome of a random experiment.

Dependent variable: the variable to be predicted in regression analysis.

Continuous probability distribution: A random variable allowed to take any value within a specified range.

Discrete random variable: A random variable that is allowed to take on only a limited number of values within the predefined range.

Expected value of random variable: The sum of the product of each value of the random variable with that values probability of happening.

Coefficient of correlation: it is the statistical measure to describe the degree of relationship of one variable with the other. It is the square root of coefficient of determination.

3.10 SELF ASSESSMENT QUESTIONS

1. State and prove the addition law of probability for any two events A and B. Rewrite the law when A and B are mutually exclusive.
2. Explain with examples the rules of Addition and Multiplication in Theory of probability.
3. Define Random variable and mathematical Expectation. How do you use the concept in a Business situation?
4. If a single draw is made from a pack of 52 cards, what is the probability of securing either an ace of spade or a jack of clubs?
5. Four cards are drawn from a full pack of cards. Find the probability that two are spades and two are hearts.
6. A bag contains 7 white and 9 black balls. Two balls are drawn in succession at random. What is the probability that one of them is white and the other is black?
7. A man is dealt 4 spade cards from an ordinary pack of 52cards. If he is given 3 more cards, find the probability p that at least one of the additional cards is also a spade.
8. Find the probability of throwing 6at least once in six throws with a single dice.
9. Six persons toss a coin turn by turn. The game is won by players who first throw head. Find the probability of success of the fourth player.
10. A person is known to hit the target in 3 out of 4 shots, where as another person is known to hit the target in 2 out of 3 shots. Find the probability of the targets being hit at all when they both try.

Notes

11. Calculate Karl Pearson's coefficient of correlation of correlation between expenditure on advertising and sales from the data given below:

Advertising expenses ("000)	65	39	62	90	82	75	25	98	36	78
Sales in lakh)	47	53	58	86	62	68	60	91	51	84

12. Obtain the equations of the two lines of regression for the data given below

X:	1	2	3	4	5	6	7	8	9
Y:	9	8	10	12	11	13	14	15	16

13. Given the following values of X and Y

X:	3	5	6	8	9	11
Y:	2	3	4	6	5	8

Find the equation of regression of i) Y on X

ii) X on Y

Interpret the result.

Notes

UNIT 4 TIME SERIES

Structure

4.0 OBJECTIVES

After going through this unit you will be able to understand:

- the meaning and concept of time series
- its application for day to day business problem
- different component of time series
- time series analysis for forecasting business demand
- mathematical forecasting models
- the utilisation of model through solved problem

4.1 INTRODUCTION OF TIME SERIES

Meaning

By a time series, we mean a series of a variable, the values of which vary according to the passage of time. In such type of variables, the time factor plays an important role in affecting the variable to a marked extent. For example, a series relating to the consumption, production or prices of certain goods, a series relating to purchase, sales, profits or losses of a certain business concern, a series relating to temperature, rainfall or yield of a particular area.

Definition

According to Croxton and Cowden, "a time series consists of data arrayed chronologically".

In the words of Ya-Lun-Chou, "A time series may be defined as a collection of readings belonging to different time periods of some economic variable or composite of variables such as production of steel, per capita income, gross national products, prices of tobacco or index of industrial production".

Utilities of Analysing a Time Series

Analysis of time series has a lot of utilities for the various fields of human interest viz. business, economics, sociology, politics, administration etc. It is also found very useful in the fields of physical and natural sciences. Some such points of its utilities are briefly mentioned below.

Notes

1. ***It helps in studying the behaviours of a variable:*** In a time series, the past data relating to a variable over a period of time are arranged in an orderly manner. By simple observation of such a series, one can understand the nature of change that take place with the variable in course of time.

2. ***It helps in forecasting:*** The analysis of a time series reveals the mode of changes in the value of a variable in course of the time. Thus, with the help of such a series we can make our future plan relating to certain measures like production, sales, profits etc. The five year plans of our country are based on the analysis of time series of the relevant data.

3. ***It helps in evaluating the performance:*** Evaluation of the actual performances with reference to the predetermined targets is highly necessary to judge the efficiency or otherwise in the progress of a certain work. For example:-the achievement of our five-year plans is evaluated by determining the annual rate of growth in the gross national product. Similarly, our policy of controlling the inflation and price rises is evaluated with the help of various price indices.

4. ***It helps in making comparative study:*** Comparative study of data relating to two or more periods, regions or industries reveals a lot of valuable information which guide a management in taking the proper course of action for the future. A time series, per se provides a scientific basis for making comparison between the two or more related set of data as in such series the data are chronologically arranged and the effects of its various components are gradually isolated and unravelled.

4.2 COMPONENTS OF A TIME SERIES

The various forces that affect the values of a phenomenon in a time series are called the components of a time series. They may be broadly classified into the following four categories:-

1. Secular trend or long term movements represented by the letter T.

2. Seasonal variations, represented by the letter S.

3. Cyclical variation, represented by letter C and

4. Irregular or random variation represented by the letter I.

The value of a variable observed at any point of time is the sum total of the above four types of components, each of which is depicted in brief as under:-

1. ***Secular trend (T):*** The word 'secular' is derived from the Latin word 'Saculum', meaning generation or age, and the word trend means the tendency of a certain things to grow, decline or to remain constant in values over a period of time. Thus, by 'secular trend', we mean the general tendency of the data to grow, decline or remain constant over a long period of time. For example, the data relating to population, production, prices, sales, income, money in circulation, bank deposits etc. have a tendency to grow in course of time; the data relating to deaths, epidemics, production capacity of a plant and machineries, value of the fixed assets etc. have a tendency to decline in course of time, and the data relating to fixed expenses depriciations, fixed income like rent, interest, etc., have a tendency to remain constant over a long period of time. This characteristic of a time series is mostly observed in the field of business and economics. A secular trend therefore, refers to the general direction and the movement of a time series considering fairly a long period of time.

2. ***Seasonal variation (S):*** Seasonal variations, in a time series, refers to those short-term fluctuations which occur regularly every season viz. yearly, half-yearly, quarterly, monthly or weekly. In the words of Prof. Hirsch, "the seasonal variation is a recurrent pattern of change within the period that results from the operation of forces connected with the climate or custom at different times to the period."

3. ***Cyclical variation (C):*** The term 'cyclical variation' refers to the recurrent variation in a time series, that usually lasts for two or more years and are regular neither in amplitude nor in length. In the words of Lincoln L Chao, "up and down movements which are different from seasonal fluctuations in that they extend over longer period of time usually two or more

Notes

years". These variations are otherwise known as oscillating movements which take place due to ups and downs recurring after a period of greater than one year. These variations, though more or less regular are not necessary, uniformly periodic.

4. ***Irregular variation:*** These variations are of irregular and indefinite pattern. They are generally mixed up with seasonal and cyclical variations and are caused by purely accidental and random factors like earthquakes, flood, famines, wars, strikes, lockouts, epidemics and revolutions etc. These variations are also otherwise called erratic, accidental, random or episodic variations. They include all types of variations in a time series which are not attributed to trend, seasonal or cyclical fluctuations. In the words of Patterson, "the irregular variation in a time series is composed of non-recurring sporadic forms which are not attributed to trend, cyclical or seasonal factors".

Characteristics of a Secular Trend

A secular trend refers to the general direction and the movement of a time series highlighted considering fairly a long period of time. The chief characteristics of a secular trend may be analysed as under:

1. ***It is either upward or downward:*** The secular trend of a series is generally either upward or downward in nature. For example, the data relating to population, production etc. have a general tendency to move upward and the data relating to birth rate and death rate due to the advancement in medical technology, improved medical facilities, better sanitation, diet etc. have a general tendency to move downward. However, such a trend may not always hold good.

2. ***It occurs as a result of such forces which are more or less stable:*** The secular trend of a series usually takes place on account of some forces which are more or less stable over a long time or which change very slowly or gradually. The examples of such forces are changes in tastes, habits and customs of people, changes in population, changes in technology, discovery of new natural sources or their depletion etc. the effects of which are very gradual, slow, smooth and move generally in one direction.

3. ***It relates to a long period of time:*** The secular trend always refers to the general tendency of the data to rise or fall over a long period of time. An evaluation of trend for a short period is not proper because there is likelihood of cyclical movement contained therein to be taken as a long period trend. Regarding the span of the period there is no such hard and fast rule. It all depends on the nature of the data under study. However, the longer, the time period of a series, the better would be the result and as a matter of safeguard, the time period should cover a minimum of two to three complete cycles.

4. ***It is likely to fluctuate round a constant:*** The secular trend of a phenomenon does not necessarily show always a rising or falling tendency. It is quite likely to fluctuate within a particular range. For example, the temperature of human body or that of a locality always fluctuates between some constant limits at various times

4.3 ANALYSIS OF A TIME SERIES

By decomposition of a time series we mean the analysis of the time series by the process of segregation of its four components viz. secular trend, seasonal variations, cyclical fluctuations and irregular movements. This involves the taking up of the following measure steps:-

1. Identification of the various factors whose interaction produces fluctuations in the time series, and

2. Measurement of the effects of these factors separately and independently by keeping the effect of the other factor constant.

Mathematical Models

For making proper decomposition of a time series we have three different types of mathematical model. They are Additive model, Multiplicative model, Mixed model

Notes

1. Additive model:- under this model, the observed value of a time series is given as $Y = T + S + C + I$
2. Multiplicative model:- under this model, the observed value of a time series is given by $Y = T \times S \times C \times I$
3. Mixed model:- under this model, an observed value of a time series is obtained by any of the following formulae based on the combination of both the additive and multiplicative models.

$$Y = TSC + I \text{ or } Y = TS + CI \text{ or } Y = T + SCI \text{ or } Y = T + S + CI \text{ or } Y = TC + SI$$

Where Y is an observed value T is trend value, S is seasonal variations, C is cyclical variations and I is irregular variation.

4.4 MEASUREMENT OF SECULAR TREND

Methods of measuring the secular trend

We have nine different methods of measuring trend values as follows:

1. Free hand graphic method.
2. Arbitrary average method.
3. Semi average method.
4. Moving average method.
5. Straight line method of least square.
6. Parabolic method of least square.
7. Geometric method of least square.
8. Exponential method of least square.
9. Growth curve method of least square.

Each of these methods listed above is explained at length as under:

1. ***Free hand graphic method:*** Under this method, the values of a time series are plotted on a graph paper in the form of a historigram. For this, the time variable is shown on the horizontal axis, the value variable on the vertical axis, and the dots are plotted on the graph paper at the intersecting points of the time and value variables. After this, a curve is drawn with free hand through the plotted dots in such a manner that it represents the general tendency of the time series and eliminates all its other components viz. seasonal, cyclical and irregular ones.
2. ***Arbitrary average method:*** Under this method, the values of any two points of time preferably, those of the extreme ones or any other close to them are arbitrarily selected as the averages of trend in the series. Basing upon these two averages, a trend line is drawn on a graph paper in a free hand smoothed manner to show the general tendency of the data. Referring to such a trend line, the trend value of the different times are determined by location.
3. ***Semi-average method:*** Under this method, the trend line is fitted to a time series basing upon the average values of its halves called semi-average. For this, the entire series is divided into two halves, leaving aside the value of the middle period, if there are odd number of periods in the series.
4. ***Moving average methods:*** Under this method, the trend line is fitted to a series on the basis of its moving averages which represent the trend value of the series. Under this method, the arithmetic averages of different groups of a set of figures are computed in a moving manner. Each group consist of equal number of items, and the first group begins with the first items and the last groups end with the last items and at each advancing step the first item of the preceding group is left aside and one more item that succeeds the group is included in the next group to get the moving average thereof.

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5. ***Straight-line method of least square:*** Under this method a straight line, called line of the best fit, is obtained by the following simple line equation:-$Y_c = a + bX$

   Where Y_c = computed trend value of Y i.e. that of the value variable

   a = intercept of Y variable i.e. the computed trend figure of Y variable when X=0

   b = slope of the trend line or the amount of change in the Y variable with reference to a change of one unit in X.

   And X = the time variable or its deviation from the time variable.

   In the above equation, a and b are the numerical constants because, for any given straight line their values do not at all change. When the values of these two constants are obtained the line of the best fit is completely determined. In order to obtain the values of these two constants (a and b) the following two normal equations are to be solved simultaneously.

   $Y = Na+bX$

   $XY = aX+bX^2$

6. ***Parabolic method of least square or polynomial fit:*** This method of least square is used only when the trend of a series is not linear but curvilinear. Under this method, a curve of parabolic type is fitted to the data to obtain their trend values and to obtain such a curve an equation of power series is determined in the following model:-

   $Y_c = a+bX+cX^2+dX^3 ....+mX^n$

7. ***Geometric or logarithmic method of least square:*** Under this method, the trend equation is obtained by $Y_c = aX^b$ Using the logarithm, the above equation is modified as under—

   $\text{Log } Y_c = \log a + b \log X$

   The above geometric curve equation should not however, be used unless there is a clear geometric progression in the value variable of a time series. Further, while using the logarithm of X, the X origin cannot be taken at the middle of the period. This limitation is overcome by the exponential trend fitting. The above trend equation can also be put in the following modified form:-

   $Y_c = aX^b + K$

8. ***Exponential method of least square:*** This method of trend fitting is restored to only when the value variable (Y) shows a geometric progression viz. 1, 2, 4, 8, 11, 32 and so on and the like. In such cases, the trend line is to be drawn on a semi logarithmic chart in the form of a straight line or a non-linear curve to show the increase or decrease of the value variable (Y) at a constant rate rather than a constant amount. When the trend takes the form of a non-linear curve on a semi-logarithmic chart an upward curve indicates the increase at varying rates depending upon the shape of the slope. The steeper the slop, the higher is the rate of increase. However, under this method, the trend line is fitted by the following model :- $Y_c = aX^b$. Using the logarithmic operation, the above equation is modified as under:-

   $Y_c = \text{Al. } (\log a+ X \log b)$

9. ***Growth curve method of least square:*** The growth curves are some special type of curves which are plotted on graph paper for analysing and estimating the trend values in the business and economic phenomena. Where initially, the growth rate is very slow but gradually it picks up at a faster rate till it reaches a point of stagnation or saturation. Such situations are quite common in business fields where new products are introduced for marketing.

Notes

4.5 MEASUREMENT OF SEASONAL VARIATION

The measures of seasonal variations are called seasonal indices which are expressed either in terms of absolute values viz. S=Y-(T+C+I) under the additive model, or in terms of percentages of the remaining components viz. $S = \frac{TSCI}{TCI} \times 100 or \frac{Y}{TCI} \times 100$ under multiplicative model. It may be noted that in order to compute the seasonal variations in a time series, the data must be expressed season wise. They cannot be computed from the data given in annual fashion for that they do not exhibit any seasonal variations in them.

There are various methods of computing the seasonal variations in a time series. The most important and popular ones among them are the following:-

1. Method of simple average.
2. Method of ratio to trend.
3. Method of ratio to moving average.
4. Method of link relatives or Pearson's method.

The procedure of each of the four methods cited as above are discussed below :

1. ***Method of simple average:*** This method is the simplest of all the methods of computing the seasonal indices in a time series. Under this method, the specific seasonal index for any particular season (month, quarter, week or day) is computed by the following model:-

$$S.I = \frac{seasonal \text{ average}}{Yearly \text{ average}} \times 100$$

   Seasonal average represents a monthly, quarterly, weekly, or a daily average which is obtained by dividing the total of a particular seasonal value by the total number of such a season in the period of analysis and yearly average represents the average of all the seasonal averages during a year.

2. ***Method of ratio to trend:*** This method is also otherwise called the percentage to trend method. As the name suggests, under this method, the trend values of a series are first determined under any of the methods of least square, and then the given data are expressed as percentage of their respective trend values. After this, the seasonal variations are found out by averaging the trend percentages by the method of mean or median. In case, the sum of the seasonal indices, thus obtained is not equal to 100% an adjustment is made to these seasonal indices by multiplying each of them by a constant correlation factor K, which is computed by:-

$$K = \frac{100\% \ value}{sum \text{ of the seasonal indices}} \times 100$$

$$\left(i.e., \frac{1200}{\sum S.I} or \frac{400}{\sum S.I} \left(in \text{ case of Qrly. Seasons} \right) \right)$$

3. ***Method of ratio to moving average:*** This is probably the most widely used method of studying seasonal variations in a time series. This method is an improvement over the 'ratio to trend' method as it tries to remove the cyclical fluctuations which remain closely mixed with the seasonal indices calculated under the 'ratio to trend method'. Under this method, the seasonal indices are calculated on the basis of the moving average trend instead of a least square trend. Further, in this method both the trends and the cyclical components are eliminated from the given data in either of the two models viz.

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a. Multiplicative model where, SI = or SI × 100 or

b. Additive model, where S+I = (T+S+C+I) – (T+C) Original value – Trend value.

4. ***Link relative method:*** This method of finding the seasonal indices in the form of the chain relatives was development by Prof. Karl Pearson and hence this method is also known as the Pearson method of seasonal variation.

4.6 FORECASTING WITH MOVING AVERAGE AND LEAST SQUARE METHOD

Meaning of Forecasting

The term forecasting means intuitive or rational estimation of the future course of events that are likely to happen in relation to a particular problem under study. Every human organization is a field of full uncertainty where the success of a person depends entirely upon his ability to forecast accurately the course of events that are likely to happen in future.

Definition

In the words of Bratt, "forecasts are statements of expected future conditions, definite statements of what will actually happen are potentially impossible. Expectations depend upon the assumptions made. If the assumptions are plausible, the forecasts have a better chance of being useful."

According to Frederick A. Ekeblad, "Forecasting refers to the use of knowledge we have at one moment of time to estimate what happens at another moment of time. The forecasting problem is created by the interval of time between the moments."

Methods of Forecasting

There are different methods of forecasting which may be broadly classified into two types:

1. Subjective Delphic method and
2. Quantitative method.

Under the quantitative method, there are different methods of forecasting but we shall only discuss the moving average and least square method.

Moving Average Method

Under this method, the trend value is fitted to a series on the basis of its moving averages which represent the trend values of the series. The moving averages are found out by calculating the arithmetic averages of different groups of a set of figures in a moving manner. Each group consists of equal number of items and the first group begins with the first item and the group ends with the last item and at each advancing step, the first item of the preceding group is left aside and one more item that succeeds the group is included in the next group to get the moving average.

Each group may consist of 2, 3, 4, 5,6, 7, 9 or 12 number of items depending on the size of the series and purpose of study. Symbolically, these are computed as under:

In case of three tons:

$$MA_1 = \sum\left(\frac{X_1 + X_2 + X_3}{3}\right), \text{ M.A}_1 = \sum\left(\frac{X_2 + X_3 + X_4}{3}\right), \text{ and so no.}$$

In case of five tons:

$$MA_1 = \sum\left(\frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}\right), \text{ M.A}_2 = \sum\left(\frac{X_2 + X_3 + X_4 + X_5 + X_6}{5}\right), \text{ and so no.}$$

Least Square Method

Under this method, a straight line, called the line of best fit, is obtained by the following simple linear equation:

$$Y_c = a + bX$$

where Y_c is the computed trend value of Y,

a is the intercept of Y, when X = 0

b is the slope of the trend line or the amount of change in the Y variable with a change of one unit in X and

X is the time variable or its deviation from the time variable.

The value of the two constants, a and b can be obtained by solving the following two normal equations:

$$\sum Y = Na + b\sum X$$
$$\sum XY = a\sum X + b\sum X^2$$

If the value of ΣX (total of deviation of time variable from its mid-value or mid-point) could be made zero, the values of the two constants, a and b, can be computed directly as under:

$$a = \frac{\sum Y}{n} \text{ and } \frac{\sum XY}{\sum X^2}$$

If the series contains even number of items, the value of each deviation (X) is to be adjusted by the following formula:

$$X = \frac{t - mid\ point\ of\ t}{1/2\ of\ the\ interval}$$

where X is the deviation of the time variable from its mid-point and t is the time variable.

Example: Forecast the trend values from the following time series using the 3 yearly moving average method.

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Production (inQuintals)	500	540	550	530	520	560	600	640	620	610	640

Solution:

Forecast of the Trend values of Production by the 3 yearly moving average method

Year	**Production**	**3 yearly**	
		Moving totals	**Moving averages**
1999	500		
2000	540 →	1590 →	530
2001	550 →	1620 →	540
2002	530 →	1600 →	533
2003	520 →	1610 →	537
2004	560 →	1680 →	560
2005	600 →	1800 →	600
2006	640 →	1860 →	620
2007	620 →	1870 →	623
2008	610 →	1870 →	623
2009	640		

Notes

Thus, the trend values of production given by the moving averages are530, 540, 533, 537, 560, 620, 623 and 623.

Example: Fit a trend line equation by the method of least square for the following data relating to the population of Odisha.

Year:	1939	1949	1959	1969	1979	1989	1999	2009
Population: (millions)	3.9	5.3	7.3	9.6	12.9	17.1	22.2	30.5

Also, forecast the population of Odisha in the year 2014.

Solution:

Here, time interval = 10, end mid point of the time variable is 1974 which has been taken as the trend origin.

1. Determination of the Straight line equations by the method of the least square.

Year t	Population Y	Time deviation i.e, $\frac{t-1974}{1/2\times 10}$ X	XY	X^2
1939	3.9	-7	-27.3	49
1949	5.3	-5	-26.5	25
1959	7.3	-3	-21.9	9
1969	9.6	-1	-9.6	1
1974	-	0	0.0	0
1979	12.9	1	12.9	1
1989	17.1	3	51.3	9
1999	22.2	5	111.0	25
2009	30.5	7	213.5	49
Total	ΣY = 108.8	ΣX = 0	ΣXY =303.4	ΣX^2 =168

Working: The straight line equation is given by $Y_c = a + bX$

Where $a = \frac{\sum Y}{N} = \frac{108.8}{8} = 13.6 \;\therefore\; \sum Y = Na + bX \;\; and \;\; EX = 0$

And $b = \frac{\sum XY}{\sum XY^2} = \frac{303.4}{168} = 1.806 \;\because\; \sum XY = a\sum X + b\sum X^2 \;\; and \;\; EX = 0$

Putting the above values of a and b in the equation the straight line equation naturalized as under:

$Y_e = 13.6 + 1.806$ X

2. Forecast of the population in the year 2014

in 2014, X = $\frac{interval - point}{1/2 \text{ time}} = \frac{2014-1974}{(1/2)\times 10} = 12$

Thus, when X = 12, $Y_c = 13.6 + 1.806 \times 12$

= 13.6 + 21.672 = 35.272 million

Hence, in 2014, the population of Odisha is expected to be 35.272 million or 35272000.

Example 1:

Sales figures of a firm is given from 2007-15 as follows.

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015
sales	40	41	45	41	50	47	45	50	46

Draw a trend line by arbitrary average method.

Solution:

Let us select the 2^{nd} year and 8^{th} year arbitrarily.

$$So,\ \text{average change} = \frac{sales\ of\ 8^{th}\ year - sales\ of\ 2^{th}\ year}{6} = \frac{50-41}{6} = 15$$

Now deduct 1.5 from the sales figure of the second year to get the trend value of the first year. Add 1.5 for each succeeding year to get the trend values of third year onwards. So various trend values will be;

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015
Trend values	39.5	41	42.5	44	45.5	47	48.5	50	51.5

Taking the trend values, now draw a trend line by using free-hand method

Example 2:

Calculate the 3yearly moving average from the following data and draw the trend line.

year	1	2	3	4	5	6	7	8	9	10	11
value	11	15	19	23	27	31	35	39	43	47	51

Solution:

Calculation of moving average

Year	(Values) y	3-yearly total	3-yearly moving average
1	11	-	-
2	15	45	15
3	19	57	19
4	23	69	23
5	27	81	27
6	31	93	31
7	35	105	35
8	39	117	39
9	43	129	43
10	47	141	47
11	51	-	-

N.B. if the period of moving average is 3 no moving average can be calculated for the first year and last year. Similarly, when period of moving average is 5 no moving average can be calculated for first two and last two years.

Example 3:

The sales of a company was Rs1,20,000 for the month of September and Rs1,30,000 for the month of October. The seasonal index was 95 for September and 115 for October. Is the increase is in accordance with the increase in seasonal index?

Notes

Solution:

When S.I. is 95, sales is Rs.1,20,000

When S.I. is 115, sales would have been $\frac{120000}{95} \times 115, 1, 45, 263$

Thus the increase in sales is less as compared to the seasonal index.

Example 4:

Year	2005	2007	2008	2009	2010	2011	2014
Production(MT)	30	40	35	42	45	38	50

Fit a straight line trend under the least square method and estimate production for 2012

Solution:

Computation of Straight Line Trend

year	**Production M.T(Y)**	**X**	**X²**	**XY**
2005	30	-4	16	-120
2007	40	-2	4	-80
2008	35	-1	1	-35
2009	42	0	0	0
2010	45	1	1	45
2011	38	2	4	76
2014	50	5	25	250
N = 7	$\sum y = 280$	$\sum x = 1$	$\sum x^2 = 51$	$\sum xy = 136$

Origin is taken as the year 2009

Since $\sum X \neq 0$, the value of a and b can be found by solving the following two equations

$\sum y = Na + b\sum X$

$\sum XY = a\sum X + b\sum X^2$

Putting the respective values as shown in the table, we get

And; 280=7a+ b(1)

136= a(1) + b(51)

Or, 7a+b= 280……..(i)

a+51b= 136……(ii)

multiplying equation (ii) by 7 and deducting eq.(i) therefrom we get;

$$\begin{array}{rcl} 7a + 357b & = & 952 \\ -7a \pm b & = & -280 \\ \hline 365b & = & 672 \end{array}$$

$$b = \frac{672}{356} = 1.89$$

putting the value of b in equation..(i) we get;

7a+1.89= 280

Notes

$$7a = 280 - 1.89 = 278.11 \text{ or } a = \frac{778.11}{7} = 39.73$$

The trend line equation will be, ye = 39.73+1.89X

For 2012 the value of X will be; 3

Production of 2012 will be 39.73+1.89(3)=39.73+5.67=45.4(MT)

Example 5 (odd number of years)

Year	2008	2009	2010	2011	2012	2013	2014
Production (`000 units)	70	80	82	75	84	90	95

1. For the above data
2. Fit a straight line trend by the method of least square.
3. Estimate production for 2015.

   Draw a graph to show the trend line.

Solution:

Computation of Straight Line Trend

year	Production (Y) (`000 unit)	X	X^2	XY	Ye
2008	70	-3	9	-210	71.91
2009	80	-2	4	-160	75.37
2010	82	-1	1	-82	78.83
2011	75	0	0	0	82.29
2012	84	1	1	84	85.75
2013	90	2	4	180	89.21
2014	95	3	9	285	92.67
N=7	$\sum Y$ =576	$\sum X$=0	$\sum X^2$ = 28	$\sum XY$=97	

The equation of the straight line is Ye = a+bX

$$\therefore \sum X = 0, \; a = \frac{\sum Y}{N} \text{ and } b = \frac{\sum XY}{\sum X^2}$$

$$a = \frac{576}{7} = 82.29 \text{ and } b = 3.46$$

So, equation of the trend line is Ye = 82.29 + 3.46X origin = 2011, X units =1 year, Y units= production in thousand units.

Calculation of trend values

At X = -3, Ye = 82.29+3.46(-3) = 71.91

X = -2, Ye = 82.29 + 3.46(-2) = 75.37

X= -1, Ye = 82.29+3.46(-1)= 78.83

X= 0, Ye = 82.29+3.46(-0) = 82.29

X=1, Ye= 82.29+ 3.46(1) = 85.75

Notes

X=2, Ye=82.29+3.46(2) = 89.21

X=3, Ye=82.29+ 3.46(3) = 92.67

For 2015, the value of X = 4 ∴ Ye = 82.29 + 3.46(4) = 96.13

4.7 SUMMARY

Time Series **–** A set of statistical data arranged in chronological order on the basis of time is termed as time series. In the analysis of time series time is the most important factor as the variable is related to time.

Secular trend - The general tendency of data to increase or decrease or stagnant over a fairly long period of time is called secular trend.

Seasonal variation – The short term fluctuation that occur regularly that too in periodic manner within a span of one year are termed as seasonal variation.

Cyclical fluctuation – The recurrent variation in time series that last longer than one year is termed as cyclical fluctuation.

Irregular variation – It is the component of the time series where the variation is totally unpredictable.

Long term variation – when variation occurs only during the long period say two to three or ten years etc.

Moving average method **–** When the prediction for future is based on most recent observations in time series, neglecting the old observations.

Short term variation **–** When variation occurs and is analysed only for few cycles of change.

4.8 SELF ASSESSMENT QUESTIONS

1. What is time series? Discuss its importance in economics and business.
2. Briefly discuss the components of a time series.
3. What is secular trend?
4. Discuss any one method of isolating trend values in a times series?
5. Draw a trend line by free-trend curve method and estimate sales for 2015.

Year	2008	2009	2010	2011	2012	2013	2014
Sale (000)	100	105	115	110	108	120	112

6. Using the arbitrary average method fit a trend line to the following data and the trend values.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014
Expenditure (000)	80	88	96	84	92	100	94	102	95

7. Calculate three yearly moving average for the following data and plot it on a graph.

year	2001	2002	2003	2004	2005	2006	2007	208	2009	2010	2011
Y	24	25	26	27	28	26	25	27	24	26	28

8. Calculate 3-yearly moving average and show it on a graph paper.

year	1	2	3	4	5	6	7	8	9	10	11
value	10	12	14	10	12	14	10	12	14	10	12

9. Use the appropriate period of moving average and calculate their values.

Notes

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
value	150	145	143	155	160	158	150	162	168	170	160	145	158	175	180

10. Fit a straight line trend by the method of least square.

Year	2005	2007	2009	2011	2013
Value	18	21	23	27	16

11. Fit a straight line trend by the method of least square.

year	2010	2011	2012	2013	2014
Sales inlakh	100	120	110	140	80

Also predict sales for 2016. Find monthly decrease.

12. Fit a straight line trend to the above data and estimate sales for 2015.

year	2003	2005	2006	2009	2011
sales	100	120	150	80	100

13. Sales figures of a firm is given from 2007-15 as follows.

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015
sales	40	41	45	41	50	47	45	50	46

Draw a trend line by arbitrary average method.

Notes

UNIT 5 LINEAR PROGRAMMING

Structure

5.0 OBJECTIVES

After going through this unit you will be able to understand:

- Meaning, characteristics and uses of linear programming.
- Linear programming model formulation procedure.
- Solution for problems through graphical methods.
- Transportation problems-loops concept and solution by north-west corner rule, least cost method and vogel's approximation method.
- Testing of optimality of the solution by modi method.

Notes

- Assignment problems and their characteristics.
- Methods of solving assignment problems.
- Comprehensive understanding through various solved examples.

5.1 INTRODUCTION

Linear programming is a widely used mathematical modeling technique to determine the optimum allocation of scarce resources among competing demands. Resources typically include raw materials, manpower, machinery, time, money and space. The technique is very powerful and found especially useful because of its application to many different types of real business problems in areas like finance, production, sales and distribution, personnel, marketing and many more areas of management.

This technique was introduced for the first time in 1947 by a Russian Mathematician George B. Dantzig.

The name 'Linear Programming' consists of the two important terms, viz., Linear and Programming. The term Linear refers to the relationship of the interrelated variables which is of the form of $y=a+bx$ where x and y are the variables of power one and a and b are the constants.

The term, programming means planning a way of action in a systematic manner with a view to achieving some desired optimal results, viz., the minimization of cost, maximization of profit etc.

Thus, Linear Programming is a Mathematical technique which is applied in the form of a linear formula for arriving at a rational proportion of the variables to be used as inputs to get the optimum result from a course of action to be planned accordingly.

5.2 DEFINITIONS

The term, Linear Programming has been defined variously by various authors. Some such definitions are quoted here as under:

According to Dantzig, Linear programming is defined as "a programming of interdependent activities in a linear structure."

According to Galton, "Linear Programming is a mathematical technique for determining the optimal solution of resources and obtaining a particular objective where there are alternative uses of resources, viz., man, material, machinery and money etc."

From the above definitions, it will be clear that linear programming is a mathematical device of ascertaining the optimal allocation of resources for obtaining the desired objective, viz., maximization of profit, or minimization of cost where various resources can be used alternatively.

5.3 APPLICATION OF LINEAR PROGRAMMING

There are three important types of problems concerning various fields where linear programming technique can be applied advantageously.

They are:

1. Problems of allocation,
2. Problems of assignment and
3. Problems of transportation.

Notes

5.4 ESSENTIALS OF LINEAR PROGRAMMING MODEL

For a given problem situation, there are certain essential conditions that need to be solved by using linear programming.

1	Limited resources	:	limited number of labour, material equipment and finance
2	Objective	:	refers to the aim to optimize (maximize the profits or minimize the costs).
3	Linearity	:	increase in labour input will have a proportionate increase in output.
4	Homogeneity	:	the products, workers' efficiency and machines are assumed to be identical.
5	Divisibility	:	it is assumed that resources and products can be divided into fractions, (in case the fractions are not possible, like production of one-third of a computer, a modification of linear programming called integer programming can be used).

5.5 PROPERTIES OF LINEAR PROGRAMMING MODEL

The following properties form the linear programming model:

1. Relationship among decision variables must be linear in nature.
2. A model must have an objective function.
3. Resource constraints are essential.
4. A model must have a non-negativity constraint.

5.6 TYPES OF LINEAR PROGRAMMING PROBLEMS

Linear Programming problems are classified into two types. They are:

1. General or Primal linear programming problem.
2. Duality linear programming problem.

The procedures of determining the desired results under the above two types of problems are contradicting to each other.

These procedures are annotated here as under

1. General or Primal Linear Programming Problem

Determination of the desired results under this type of problem will involve the following steps:

Step No. 1. Formulation of the Given Problem.

Step No. 2. Solution of the Formulated Problem.

Each of the above steps will again require the following substeps. Formulation of L.P.

Problem Under this step the given data relating to a problem will be arranged first under the following three types of equation functions.

1. Objective function,
2. Constraint functions, and
3. Non-negative functions.

1. ***Objective Function***. The objective of a problem may be either to maximize or minimize some result. If it is a case of profit, income, or output the objective must be maximization. But if it

Notes

is a case of loss, cost or input, the objective must be minimization. For this, the rate of profit or cost per variable in issue must be assessed first and then the number of each variable will be represented in the function through some letters viz. X, Y to be ascertained through the process of solution. As such the objective function will be presented in the following form:

$Z_{(P)} = P_1X_1+P_2X_2+.....+ P_nX_n$

$Z_{(P)}$ = Maximum amount of profit

where, P_1, P_2, ….P_n = Rate of profit per different variables to be produced, viz. goods or services.

X_1, X_2,….. X_n = The number of different variables to be produced under the decision.

In case of variables involving cost or loss, the objective will be minimization and in that case the objective function will be formulated as under:

$Z_{(C)} = C_1X_1 + C_2X_2 +... + C_nX_n$

where, $Z_{(C)}$ = Minimum amount of cost

C_1, C_2,……., Cn = Cost per unit of the variable

X_1, X_2, ……., X_n = Different number of the different variables.

2. ***Constraint Functions.*** To accomplish the desired objective it is necessary to put some resources, viz, manpower, material, machine or money in the process of: production or performance. But such resources may not be available in unlimited quantity, in all the cases. There are some resources which may be available to a limited extent and thus create constraints or bottlenecks in the process of performance. There are also some resources whose availability cannot be obtained below a certain extent and thus compels the management to procure them in certain larger quantities. However, there might be some resources, which may be available to the extent just required for the purpose. These resources, therefore, do not create any obstacle. But the resources which are available up to or beyond certain limit create constraints in achieving the objective. Thus, the objective function will be adjusted in the light of the given constraints relating to the availability of the various resources. Hence, the objective function will be followed by the constraint functions in the following manner.

Constraints of the

Process—I : $a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$

Process—II: $a_{21}X_1 + a_{22}X_2 + ... + a2_nX_n \geq b_2$

Process—III: $a_{31}X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$

Here, a represents the quantity of a particular resources required in a particular process, and b the total of quantity of the resources available for a process.

3. ***Non-Negative Function.*** This function implies that the production or performance of the variables in issue will never be negative. It will be either zero or greater than zero but never less than zero. This function is therefore represented as Sunder:

$X_1 \geq 0, X_2 \geq 0 ,.., X_n \geq 0$

Thus, the formulation of a primal linear programming problem will be constituted as below:

Mathematical Formulation of the Linear Programming Problems

(Primal)

Maximize or Minimize $Z = C_1X_1 + C_2X_2 +... + C_nX_n$

Subject to the constraints:

$a_{11}X_1+a_{12}X_2 +... + a_{1n}X_n \leq b_1$

$a_{21}X_1+a_{22}X_2 +…… + a_{2n}X_n \geq b_2$

Notes

$a_{3l}X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$

Subject to the non-negativity condition that

$X_1 \geq 0, X_2 \geq 0, ..., X_n \geq 0$

Alternatively,

Determine the real numbers X_1, X_2 ... and X_n such that

$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$

$a_{2l} X_l + a_{22}X_2 + ... + a_{2n}X_n \geq b_3$

$a_{31} X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$

$X_1, X_2, X_n \geq 0$

And for which expression (Objective Function)

$Z = C_1X_1 + C_2X_2 + ... C_nX_n$ may be the maximum or minimum.

2. Dual Problem

The transpose of a primal problem of liner programming is known as a Dual Problem. To reduce the computational procedure under the simplex method, every primal problem of LPP can be converted into a Dual problem. This is suitable particularly when the primal problem involves lesser number of variables (unknown) than the number of constraints (restrictions.)

Procedure of Formulation

From a given primal problem the Dual Problem is derived by the process of transposition which involves the following steps:

- Formulate the given problem first in the form of a primal problem.
- Convert the objective symbol is to Z*
- If the given objective is maximization, count the same into minimization and vice-versa.
- Convert the given variable X_1, X_2, X_3, … etc. in to Y_1, Y_2, Y_3, … etc. respectively.
- Convert the coefficients of the objective functions into the constants of the constraint functions respectively. Also convert the constants of the constraints into the coefficients of the objective function respectively.
- Convert the columns of coefficients of the constraints in the primal problem into the rows of coefficient of the constraints in the Dual Problem respectively.
- Convert the $\leq$ signs of the constraints into the $\geq$ signs of the constraints and vice-versa without any change in the $\geq$ sign of the non-negative conditions and = and ≅sign of the constraints. By the procedure depicted above the dual problem will be brought to the shape of a formulation.

Procedure of Solution

The procedure of solving a Dual problem under both the graphic and the simplex method of solution will remain the same respectively as in case of a primal problem.

Practical Application of L.P. Problems and its Formulation

1. ***Optimal Product line Problem.*** An optimal product line problem is one which needs decision as to how much of 'n' different products should a firm produce or sell when each of the products requires a particular proportion of various resources, viz. material, labour, machine hour etc. the supply of which are limited to a certain extent.

Illustration: **(Product line Problem)**

A firm produces two types of products P and Q through two processes, viz, Foundry and Machine shop. The number of man-hours required for each unit of P and Q in each of the processes and the number of man-hours that can be availed at best in the two processes are given as follows.

Notes

	Foundry process	Machine process
Product P	10 units	5 units
Product Q	6 units	4 units
Available at best	1000 units	600 units

Net profit expected from each unit of the product is: P—Rs. 50 and Q—Rs. 40. Formulate the problem for solution to arrive at the optimal number of the two products, P and Q to be produced.

Solution:

Here the problem obviously involves the maximization of profits. Thus, the formulation of the problem will be made in the following order:

Step-I: Notation

Let Z = Total of maximum possible Net profit

X_1 = No. of the product, P to be produced

X_2 = No. of the product, Q to be produced

F = Foundry process;

And M = Machine shop process.

Step-II: Decision Table

At this step, the given data will be arranged in a table in the following order:

Product	Decision variable	F (process) units of man hour	M (process) units of man hour	Net profit per unit Rs.
P	X_1	10	5	50
Q	X_2	6	4	40
Maximum Labour hours available		1000	600 units	

Step-III: Constitution of the different linear functions.

1. Objective function

   Maximize profit $Z = 50X_1 + 40X_2$

2. Constraint functions

   (a) Foundry constraints, $10X_1 + 6X_2 \leq 1000$

   (b) Machine shop constraints, $5X_1 + 4X_2 \leq 600$

3. Non-negative functions: $X_1 \geq 0$, $X_2 \geq 0$

Step-IV: Formulation of L.P.P.

Determine the real numbers X_1 and X_2 such that

$10X_1 + 6X_2 \leq 1000$

$5X_1 + 4X_2 \leq 600$

Notes

$X_1, X_2 \geq 0$

And for which the objective function, $Z = 50X_1 + 40X_2$ may be a maximum.

2. ***Diet Problem.*** A diet problem is one in which a decision is taken as to how much of 'n' different foods to be included in a diet given the cost of each food and the particular combination of nutrient each food contains. Here, the objective is to minimize the cost of diet such that it contains a certain minimum amount of each nutrient.

Illustration: A poultry firm contemplates to procure four special feeds in a combination which would provide the required vitamin contents and minimize the cost as well. From the following data formulate the linear programming problem.

Feed	**Units of vitamins A, B, C, in each feed**			**Feed cost Rs.**
	A	B	C	
P	4	1	0	2
Q	6	1	2	5
R	1	7	1	6
S	2	5	3	8

Minimum vitamin contents needed per feed mix in units

A—12

B—14

C—8

Solution: Step-I: Notations

Let Z = Total minimum possible costs

X_1 = Decision variable for the feed, P

X_2 = Decision variable for the feed, Q

X_3 = Decision variable for the feed, R

X_4 = Decision variable for the feed, S

Step-II: Decision Table

Feed	**Decision Variable**	**Units of vitamins A, B, C, in each feed**			**Feed cost Rs.**
		A	B	C	
P	X_1	4	1	0	2
Q	X_2	6	1	2	5
R	X_3	1	7	1	6
S	X_4	2	5	3	8
Minimum needed		12	14	8	

Step-III : Constitution of the different Linear Functions

1. Objective Function

   Minimize cost,

   $Z = 2X_1 + 5X_2 + 6X_3 + 8X_4$

Notes

2. Constraint Functions:

   Vitamin A constraints,

   $4X_1 + 6X_2 + X_3 + 2X_4 \geq 12$

   Vitamin B constraints,

   $X_1+X_2 + 7X_3 + 5X_4 \geq 14$

   Vitamin C constraints,

   $0X_1+2X_2 + X_3 +3X_4 \geq 8$

3. Non-negative Functions :

   $X_1, X_2, X_3, X_4 \geq 0$

Step-IV : Formulation of the L.P.P.

(Minimize cost) $Z=2X_1 + 5X_2 + 6X_3 + 8X_4$

Subject to the constraints

$4X_1+6X_2 + X_3 + 2X_4 \geq 12$

$X_1+X_2+7X_3 + 5X_4 \geq 14$

$0X_1 + 2X_2 + X_3 + 3X_4 \geq 8$

And subject to the non-negativity condition that

$X_1, X_2, X_3, X_4 \geq 0$

5.8 SOLUTION OF THE FORMULATED PROBLEMS

After a linear programming problem has been properly formulated, the next step is to attempt on its solution to determine the values of the different decisive variables, viz. X_1, X_2, X_3 etc. depicted in the formulation of the said linear programming problem.

Solution of the L.P.P. may be of three types, viz.

Feasible solution,

Non-feasible solution, and

Optimal solution.

1. ***Feasible Solution.*** A solution which satisfies the non-negativity conditions of a general L.P.P. is called a feasible solution.
2. ***Non-feasible Solution.*** A process of solution which does not satisfy the lion-negativity conditions of a general L.P.P. is called a Non-feasible solution.
3. ***Optimal Solution.*** A feasible solution which optimizes (minimizes or maximizes) the objective function of a general L.P.P. is called an optimum solution.

5.9 METHODS OF SOLUTION OF L.P.P.

There are two different methods of solving a linear programming problem.

(1) Graphic Method and (2) Simplex Method

1. Graphic method

A linear programming problem which involves only two decisive variables, viz. X_1 and X_2 can be easily solved by Graphic method. But a problem which involves more than two decisive variables cannot be solved by the Graphic Method. This is because, a graph ordinarily has two axes only, i.e., horizontal and vertical and thus, more than two decisive variables cannot be represented and solved through a graph.

Procedure of Graphic Solution. For solution of a L.P.P. through a graph the following steps are to be taken up one after another:

1st Step: Formulation of the L.P.P.

2nd Step: Conversion of the constraints functions into the equations and determination of the values of each of the variables under each equation by assuming the other variable to be zero.

3rd Step: Drawal of the 1st quadrants of the graph in which only positive values of both the variables are plotted on the basis of the non-negativity condition, i.e. $X_1, X_2 \geq 0$.

4th Step: Plotting of each set of points on the graph for the pair of values obtained under each of the equations and joining them differently by straight lines.

5th Step: Identification of the feasible region through shaded area which satisfies all the constraints. For "less than or equal to constraints" such region will lie below all the constraint lines but for "greater than or equal to constraints" the said region will lie above all the constraint lines.

6th Step: Location of the corner points or the extreme points of the feasible region.

7th Step: Evaluation of the objective function at each of the corner points through the following table.

Evaluation Table

Corner Points	**Values of the Variables**	**Objective Function**	**Total of Values of Z**
	X_1 X_2	$Z = RX_1 + RX_2$	Rs

If the objective function relates to maximization, the corner point showing the maximum value in the above table will give the optimal solution for the values of X_1 and X_2. On the other hand, if the objective function relates to minimization, the corner point showing the minimum value in the above table will give the optimal solution for the values of X_1 and X_2.

Alternative Method of Evaluation.

1. Plot a line for the objective function assuming any value for it so that it falls within the shaded area of the graph. Such line is known as iso-profit or iso-cost line.
2. Move this iso-profit (iso-cost) line parallel to itself and farther (closer) from (to) the origin till it goes completely outside the feasible region.
3. Identify the optimal solution as the co-ordinates of that point on the feasible region touched by the highest possible profit line (lowest possible cost line).
4. Read the optimal co-ordinates of X_1 and X_2 from the graph and compute the profit or the cost.

On Maximization.

ILLUSTRATION A firm proposes to purchase fans and sewing machines. It has only Rs. 5760 to invest and space for at most 20 items. A fan costs Rs.360 and a sewing machine Rs. 240, Profit expected from a fan is Rs.22 and a sewing machine is Rs.18. Using the graphic method of solution determine the number of fans and sewing machines he should purchase to maximize his profit. Also ascertain the maximum possible profit he can earn.

Solution:

Step-I : Formulation of the Problem.

Formulation of the problem can be made directly by the following two steps.

Articles	**Decision Variables**	**Constraints**	**Profit per unit Rs.**

Notes

		Investment	Space	
Fan	X_1	360	1	22
Sewing Machine	X_2	240	1	18
Maximum Capacity		5760	20	

2. Formulation of L.P.P.

Maximize (Profit) $Z = 22X_1 + 18X_2$

Subject to Constraints :

1. $360X_1 + 240X_2 \leq 5760$
2. $X_1 + X_2 \leq 20$ and

Subject to non-negativity condition

$X_1, X_2 \geq 0$

Step II: Conversion of the constraints into equations and determination of the values of the ordinates.

1. $360X_1 + 240X_2 = 5760$

   Here, when $X_1 = 0$ $X_2 = 24$

   when $X_2 = 0$ $X_1 = 16$

2. $X_1+X_2 = 20$

   Here, when $X_1 = 0$ $X_2 = 20$

   when $X_2 = 0$ $X_1 = 20$

Steps III to VI:

Graphic Representation of the Constraints lines of the L.P.P.

Step VII: Evaluation of the Objective Function.

Corner Points	**Values of the Co-ordinates**		**Objective Function $22X_1 + 18X_2$**	**Total of Values of Z Rs.**
	X_1	X_2		
P	0	0	22 + 18x0 =	0
Q	0	20	22 x 0 + 18 x 20 =	360
R	8	12	22x8+18x12 =	392 (Max)
S	16	0	22x16+18x0 =	352

Hence, the company should purchase 8 units of X_1 i.e. Fan and 12 units of X_2 i.e. sewing machine to make the maximum profit of Rs. 392.

On Minimization

Illustration: A firm produces three different products, viz., R, S, and T through two different plants, viz., P_1 and P_2, the capacities of which in number of products per day are as follows:

Plant	**Product R**	**Product S**	**Product T**
P_1	3,000	1,000	2,000
P_2	1,000	1,000	6,000

Notes

The operating cost per day of running the plants P_1 and P_2 are Rs.600 and Rs.400 respectively. The expected minimum demands during any month for the products R, S and T are 24000 units, 16000 units and 4800 units respectively. Show, by Graphic method how many days should the firm run each plant during a month so that the production cost is minimized while still meeting the market demand.

Solution: **1. Notation**

Let Z = objective function (Minimization of cost)

X_1, X_2 = number of working days of the plants P_1 and P_2 respectively.

2. Decision Table

Name of plants	Decision variables	Constraints of the Products			Cost per day Rs.
		R units	S units	T units	
P_1	X_1	3000	1000	2000	600
P_2	X_2	1000	1000	6000	400
(Minimum demands		24000	16000	48000	

3. Formulation of L.P.P.

Objective Function:

Z (Minimize) = $600\ X_1 + 400X_2$

Subject to

$3000\ X_1 + 1000\ X_2 \geq 24000$

$1000\ X_1 + 1000\ X_2 \geq 16000$

$2000\ X_1 + 6000\ X_2 \geq 48000$

And $X_1, X_2 \geq 0$

4. Conversion of the constraints into equations and determination of the values of the different sets of ordinates.

1. $3000\ X_1 + 1000\ X_2 = 24000$

   Let $X_1 = 0$, then $X_2 = 24$

   $X_2 = 0$ then $X_1 = 8$

2. $1000\ X_1 + 1000\ X_2 = 16000$

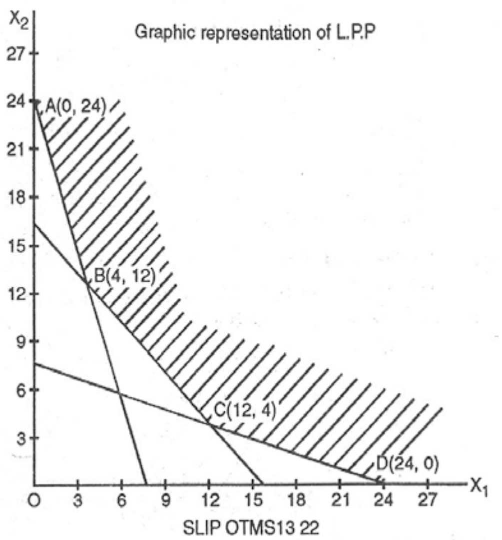
   Let $X_2 = 0$ then $X_2 = 16$

   $X_2 = 0$ then $X_1 = 16$

3. $2000\ X_1 + 6000\ X_2 = 48000$

   Let $X_1 = 0$ then $X_2 = 8$

   $X_2 = 0$ then $X_1 = 24$



6. Evaluation of the Objective Function by

Corner	Co-ordinates	Objective Function	Total of Z Rs.

Notes

points	X_1	X_2	$Z = 600X_1 + 400X_2$	
A	0	24	600 x 0 + 400 x 24	9600
B	4	12	600 x 4 + 400 x 12	7200 (Min.)
C	12	4	600 x 12 + 400 x 4	8800
D	24	0	600 x 24 + 400 x 0	14400

From the above evaluation table it comes out that the optimal solution lies at the corner point B, where the total cost is the minimum, i.e., Rs. 7200. Hence, the firm should run the plant I, for 4 days and plant II for 12 days to minimize its cost and to meet with the expected demand for the market as well.

5.10 TRANSPORTATION PROBLEMS

MEANING AND CHARACTERISTICS OF A TRANSPORTATION PROBLEM

Meaning

It is a special type of linear programming problem which deals with the transportation of certain homogeneous goods, or services from their different sources of origin to their different destination of requirements viz. from factories to warehouses, or from warehouses to stores etc. The chief objective of such type of problem is to minimize the cost or maximize the revenue of transportation by satisfying the requirements of the different destinations within the given constraints of their different sources of supply. The data relating to such type of problems are usually tabulated as under.

Transportation Table

(Cost or Revenue in rupees)

Destination (To) / Sources (From)	Ware House 1	Ware House 2	Ware House 3	Supply Available
Factory 1	@Rs.	@Rs.	@Rs.	No. of units
Factory 2	@ Rs.	@ Rs.	@Rs.	No. of units
Factory 3	@ Rs.	@ Rs.	@ Rs.	No. of units
Factory 4	@Rs.	@Rs.	@ Rs.	No. of units
Requirements (in units)	No. of units	No. of units	No. of units	Total No. of units

The total number of units available at the various sources may, or may not be equal to the total number of units demanded by the various destinations. If these two totals tally, it is called a case of balanced distribution, else, it is called a case of unbalanced distribution.

The method applied in solving this type of linear programming problems is popularly known as the Transportation method, Distribution method, or the Method of Allocation. This method is nothing but a simplified version of the simplex method of linear programming problem. It was originally introduced by F.L.Hitchcock in 1941 in his work "the distribution of a product from several sources to numerous locations". Subsequently it was developed by T.C. Koopan in 1947 in his work "Optimal utilisation of transportation system" and by Dantzig in 1951 in the formulation and solution of linear programming problems.

Notes

Characteristics

The chief characteristics of a transportation problem may be outlined as under.

1. This problem involves the optimization of the objectives of a person, or a management.
2. All the units available are assigned to the various demanding centres (destinations), and all the units required by the different destinations are supplied by the different sources of supply. In case of shortage of either of the two types of centres, the same is made up by introduction of the dummy variables.
3. The availability as well as the requirements of the various centres (i.e. sources and destinations) are finite and constitute the limited resources.
4. The cost, or the revenue of transportation is linear (i.e., the cost or revenue of objects is n times the cost or revenue of a single object).
5. It involves a large number of variables, and linear constraints for which it needs a special method of solution and the simplex method of solution is felt more tedious and time taking even with the help of electronic computers.
6. It extends to a variety of problems i.e. scheduling, investment, production, personnel and allotment etc.

Types of transportation problem

Broadly speaking, transportation problems are of two types. They are: (1) Balanced Problems and (2) Unbalanced Problems.

1. Balanced Problems

A transportation problem is said to be a balanced one when the total number of units available at various sources is equal to the total number of units required by the various destinations. The following table exhibits the example of a balanced transportation problem.

Table 5.1: Showing a Balanced Transportation Problem

From	Store A	Store B	Store C	Units Available
Ware House I	@ Rs.	@Rs.	@ Rs.	60
Ware House II	@Rs.	@Rs.	@Rs.	30
Ware House III	@ Rs.	@Rs.	@Rs.	10
Units Required	15	25	60	100

From the above table, it must be seen that the total number of units available at the three warehouses is 100 and the total number of units required by the three stores is also 100. Hence, it is a case of balanced transportation problem.

2. Unbalanced Problem

A transportation problem is said to be an unbalanced one, when the total number of units available at the different sources are either more or less than the total number of units required by the different destination centres. For example, if the total number of units available is 100 and the total number of units required is 120 or alternatively, they are 200 and 150 respectively, it is a case of unbalanced transportation problem. In such a case, before attempting on the solution, the given unbalanced problem is first balanced by introduction of a dummy variable either for the source or for the destination centre as the case may be. The following tables show the example of the unbalanced transportation problems and the way of their balancing.

TABLE—II

Notes

(Showing an Unbalanced Transportation Problem with Excessive Supply)

Warehouse / Plant	W_1	W_2	W_3	W_4	Units Available (Supply)
Plant I	@Rs.	@ Rs.	@ Rs.	@Rs.	70
Plant II	@ Rs.	@Rs.	@ Rs.	@Rs.	60
Plant III	@ Rs.	@ Rs.	@Rs.	@ Rs.	20
Units Required (Demand)	20	30	40	50	140/150

TABLE—III

(Showing the Balancing of the Above Unbalanced Transportation Problem)

Plant / Warehouse	W_1	W_2	W_3	W_4	Dummy Warehouse	Units Available
Plant I	@	@	@	@	0 0	70
Plant II	@	@	@	@	@0	60
Plant III	@	@	@	@	@0	20
Units Required	20	30	40	50	10	150

TABLE—IV

(Showing an Unbalanced Transportation Problem with Excessive Demand)

Stores / Warehouse	S_1	S_2	S_3	Units Available
W_1	@	@	@	10
W_2	@	@	@	20
W_3	@	@	@	30
W_4	@	@	@	10
Units Required	20	30	40	90/70

TABLE—V

(Showing the balancing of the above unbalanced transportation problem)

Stores / Warehouse	S1	S2	S3	Units Available
W1	@	@	@	10
W2	@	@	@	20

Notes

W3	@	@	@	30
W4	@	@	@	10
Dummy Warehouse	@ 0	@0	@ 0	20
Units Required	20	30	40	90

5.12 DEFINITION OF CERTAIN TERMS

Certain terms peculiar to a transportation problem are defined here as under :

1. ***Source.*** It refers to a place of origin from which the goods or services are supplied to the different places of destination. It includes a factory, a plant, a warehouse, etc. which are arranged in different rows in a table.

2. ***Destination.*** It refers to a place of demand where the goods or services are required in certain finite quantities. It includes a warehouse, a store or a market which are arranged under different columns in a table.

3. ***Cell.*** It refers to a square in a transportation table where the corresponding rate of cost or revenue per unit of goods transported from a source to a destination is put.

4. ***Feasible solution.*** It refers to a set of non-negative individual allocations of goods and services that simultaneously removes deficiencies.

5. ***Basic feasible solution.*** It refers to a feasible solution to a transportation problem with 'm' sources and 'n' destinations in which the number of positive allocations is one less than the sum of rows and columns.

6. ***Degenerate basic feasible solution.*** It refers to a feasible solution to a transportation problem with 'm' sources and 'n' destinations in which the number of positive allocations is less than the sum of the rows and column minus one i.e. number of cells allocated

   i.e.: $< (m + n — 1)$.

7. ***Optimal solution.*** It refers to a feasible solution that optimizes the objective: (i.e., minimizes the cost or maximizes the revenue) of a transportation problem.

8. ***Stone.*** It refers to the positive allocation made of the goods, or services in particular cell or square of the table. It is usually covered by a circle viz :

   (10) (25) (30), etc.

9. ***Stone square.*** It refers to a square or a cell in a table that contains some positive allocations or stones.

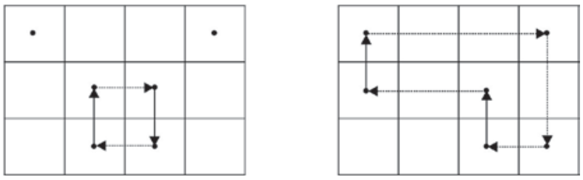
10. ***Water square.*** It refers to an empty square, or a cell that does not contain any positive allocations or stones.

11. ***Home square.*** It refers to the water square in which the difference between the values of (R + C) of the water square and the cost of the concerned water square is the highest, (highest positive in case of minimization and highest negative in case of maximization). Here, R refers to the row and C refers to the column of a water square.

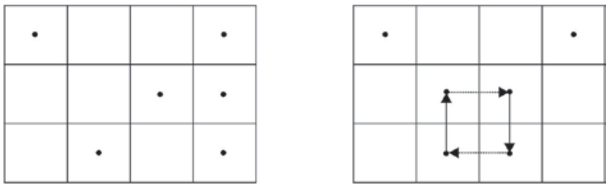
12. ***Path.*** It refers to a chain of arrow lines that starts from the home square and runs across the stone squares turning at each step at a right angle and returns back to the home square with the fewest possible steps. In this path, the first step is given + sign, the second step — sign, and so on in an alternative manner and the step back to home square is always given a — sign.

Notes

13. ***Loop.*** It consists of horizontal and vertical lines with an allocation at each corner, which in turn, is a joint of horizontal and vertical line. In other words, it is an ordered set of four or more cells in which any two adjacent cells lie either in the same row or in the same column. A feasible solution to a transportation problem is basic if and only if the corresponding cell in the transportation table does not contain a loop. The following diagrams show the examples of loops.



14. ***Independent Position.*** It refers to a set of allocations for which it is impossible to form any closed loop. The following diagrams exhibit the examples of Independent and non-independent positions.



5.13 TYPES OF SOLUTION

There are two types of solution of a transportation problem. They are :

(l) Basic solution and (2) Optimal solution.

1. Basic Solution. A basic solution to a transportation problem is said to have arrived at, when the number of positive allocations, i.e., stones in the table is equal to (m + n 1) notwithstanding the optimization of the cost or revenue involved.

2. Optimal Solution. As pointed out earlier, an optimal solution to a transportation problem is said to have arrived at, when the number of positive allocations i.e., stones is equal to the sum of the number of rows and columns minus one i.e.,

   n (stones) = (m + n — 1) and the objective of the problem is optimized i.e., the cost is minimized or the revenue is maximized).

5.14 DIFFERENT METHODS OF BASIC SOLUTION

There are three different methods of basic solution of a transportation problem. They are:

1. North-West Corner Method (NWCM)

2. Least-Cost Method (LCM)

3. Vogel’s Approximate Method (VAM)

The results obtained under these different methods are, however, likely to be different. The nature of each of these methods is explained as under.

5.14.1 North-West Corner Method (NWCM)

The procedure of this method is briefly outlined as under.

1. Construct first, a table with required number of rows and columns to accommodate all the sources (plants, factories etc.) and all the destinations (warehouses, markets, stores etc.) respectively with or without their respective cost figures.
2. Allocate the resources starting with those available at the source 1 in the first row. The allocations are to be made beginning with the cell (1, 1) at the north-west corner (i.e., the

Notes

upper left hand corner) in such a way that either the units available at the source concerned or the units required by the destination concerned is exhausted. If the units of a source are exhausted then the next source should be utilized to allocate the balance required in the destination concerned. On the other hand, if the destination is exhausted leaving the surplus with a source, the same surplus should be allocated to the next destination lying on the same row.

3. Go on allocating the resources in the above manner until all the rim requirements... are satisfied i.e. all the sources are exhausted beginning with the upper left hand corner and ending with the lower-right hand corner of the transportation table.
4. Compute the costs of transportation or revenue earned there from by multiplying the allocated figures (stones) with their corresponding costs (revenue) and get them totalled.

Example 1: Determine the basic feasible solution to the following transportation problem using the north-west corner rules.

Cost Matrix

To / From	W_1	W_2	W_3	W_4	Units Available
F_1	6	4	1	5	14
F_2	8	9	2	7	16
F_3	4	3	6	2	5
Units Required	6	10	15	4	35

Solution:

Basic Solution of the Transportation Problem by the North-West Corner Method

Warehouse / Factory	W_1	W_2	W_3	W_4	Factory Capacity
F_1	6 (cost 6)	8 (cost 4)	(cost 1)	(cost 5)	14
F_2	(cost 8)	2 (cost 9)	14 (cost 2)	(cost 7)	16
F_3	(cost 4)	(cost 3)	1 (cost 6)	4 (cost 2)	5
Warehouse Requirement	6	10	15		35

Notes:

1. The given cost figures are shown at the upper-right corner of each cell and the allocated figures are shown within the circles called stones.
2. The total cost of transportation in the given problem is computed as under:

   (6x6) + (8x4) + (2x9) + (14x2) + (1x6) + (4x2)

   =36 + 32 + 18 + 28 + 6 + 8 = 128
3. The above table shows that the number of stones (i.e. the cells in which allocations fare made) is 6 which equals m + n — 1 (i.e., 4 + 3 — 1). Thus, the basic solution is found to have been arrived at with the above allocations.

5.14.2 Least Cost Method (LCM)

This method is also otherwise called the Lowest cost entry Method, or the Matrix Minima Method.

Notes

The procedure of the method is outlined as under :

1. Construct first, a table .with the required number of rows and columns to accommodate all the sources (plant, factories etc.) and the destinations (warehouse, markets, stores etc.) respectively, along with their respective cost figures.
2. Allocate the resources starting with the cell of the lowest possible cost until either its source capacity (total of its row) or the destination requirement (total of its column) is exhausted. Then move to the cell with the next lowest cost, and allocate the resources until either its respective source capacity (total of its row) or destination requirement (total of its column) is exhausted. Cross off the remaining cells of the same row, or the same column, the total of which is exhausted by the allocation of the resources to any of the cells lying on it.
3. Go on allocating the resources in the above fashion until all the rim-requirements are satisfied (i.e., all the sources are exhausted allocating in order of the lowest cost). In case, there arises a tie for the lowest cost cells during any allocation (i.e. when more than one cell contains the same lowest cost), choose any one of these cells arbitrarily for allocation.
4. Compute the cost of transportation by multiplying the respective allocated figures with their corresponding costs and get them totalled.

   Example 2: From the data given in the example 1 above, determine the basic solution to the transportation problem using the least cost method.

Solution:

Basic solution of the transportation problem by the least cost method

Warehouse / Factory	W_1	W_2	W_3	W_4	Factory Capacity
F_1	6 x	4 x	1 *1 14	5 x	14
F_2	8 *5 6	9 *6 9	2 *2 1	7 x	16
F_3	4 X	3 *4 1	6 x	2 *3 4	5
Warehouse Requirement	6	10	15	4	35

Notes:

1. The asterick marks *1, *2 etc. indicate the allocations made in order. Here, we have started allocating with the cell (1, 3) as the cost 1—put here is the least possible, and then go through the cells (2, 3), (3, 4), (3, 2), (2, 1), and (2, 2) successively, in order of the least costs.
2. In case of the two cells, (2, 3) and (3, 4) containing the same cost figure 2, there is a tie position for which we have selected the cell (2, 3) arbitrarily for allocation.
3. The cells bearing the costs 4, 5, 6, and 7 could not be allocated as they were already crossed by the previous exhaustive allocations in the respective columns, or rows.
4. The basic solution is found to have arrived at, as the number of stones, n (s). = m + n — 1 i.e. 4 + 3 — 1 = 6.

Computation of the total cost

(14 x 1) + (6 x8) + (9 x 9) + (1 x 2) + (1 x 3) + (4 x 2)

Notes

= 14 + 48 + 81 + 2 + 3 + 8 = 156.

It must be seen that this cost of Rs. 156 is more than the cost of Rs. 128 arrived at before, under the North-West corner method. This is due to the fact that the cost-rates shown in the table are at higher rates for the nearby warehouses than those for the distant warehouses.

Hence, by the lowest cost entry method, it should not be meant that it will always produce the lowest total cost of transportation. It may give more or less total cost than any of the other two methods depending upon the cost data given.

5.14.3 Vogel's Approximation Method (YAM)

This method is also otherwise known as the Unit Cost Penalty Method (Unit Revenue Reward Methods in case of maximization).

Here, penalty (reward) refers to the difference between the two best costs (best-revenue) in each row and column of the table. This difference is considered as penalty (reward) for making allocations in the second lowest cost (second highest revenue) entries instead of the lowest cost (highest reward) entries in each row or column.

However, the detailed procedure of allocating the goods or services under this method is outlined as under.

1. Construct the initial transportation table with the cost (revenue) data as given with the required number of rows and columns to accommodate all the sources and destinations.
2. Find the difference between the two best costs (revenue) for each of the rows and columns as the penalty (rewards) and put them at the cost revenue column and row respectively.
3. Allocate the maximum possible quantity to the cells with the lowest costs (highest revenue) in order against the maximum penalty (minimum reward) columns or row and cross off the column or row the total value of which is exhausted. In deciding the maximum quantity to be allocated to a cell, the total of its row value, or column value whichever is less should be taken into account. In case of a tie in regard to the largest penalty (smallest reward), select any one of them arbitrarily.
4. After the allocation in a row (column) is over, the penalties (rewards) shall be revised again for the next allocation after excluding the column (row) crossed off already. Then make the allocation in the next step in accordance with the step 3 explained above till the basic solution is arrived at (i.e. the number of stones is equal to the sum of the rows and columns minus one i.e., (in + n — 1).
5. Then compute the cost (revenue) of each individual allocation with reference to its corresponding cost and find the total of the transportation cost by the process of addition.

   The following examples will illustrate how the basic solution to a transportation problem is worked out under this method.

   Example 3: From the data given in the example 1 above, find the basic solution to the transportation problem by Vogel's Approximation method. SOLUTION:

TABLE—I

Basic solution of the transportation problem under the VAM

Warehouse / Factory	W_1	W_2	W_3 X	W_4	Units Available	Penalty	Index
F_1	6	4	1	5	14	3	
F_2 X	8	9	2 15	7 1	16	5	← Largest Penalty

Notes

F_3	4	3	6	2	5	1	
Requirements	6	10	15	4	35		
Penalty	2	1	1	3			

TABLE—II

Excluding the column 3 and row 2 crossed off as above

Warehouse / Factory	W_1 X	W_2	W_4 X	Net Available	Penalty
F_1	6	4	5	14	1
F_3 X	4	3	2 3	5	1
Requirement	6	10	3	19	
Penalty	2	1	3		
Index			↑ Largest Penalty		

TABLE—III Excluding the Column 4 Crossed off

Warehouse / Factory	W_1	W_2	Net Available	Penalty	
F_1	6 4	4 10	14	2	← largest penalty opted
F_3	4 2	3	2	1	
Requirement	6	10	16		
Penalty	2	1			

From the above three tables, it is seen that there are 6 allocations (stones) which is equal to m + n — 1 (i.e. 3 + 4 — 1).

Thus, the basic feasible solution is found to have been arrived at with the allocations as under:

From F_2 to W_3 =	15	Units
F_2 to W_4 =	1	Unit
F_3 to W_4 =	3	Units
F_1 to W_1 =	4	Units
F_1 to W_2 =	10	Units
F_3 to W_1 =	2	Units
Total	35	Units

Computation of the cost :

Notes

The cost of transportation in the above case is computed as under.

(15 x 2) + (1 x 7) + (3 x 2) + (4 x 6) + (10 x 4) + (2 x 4)

= 30 + 7 + 6 + 24 + 40 + 8 = 115

This cost of Rs. 115 is much less than the cost of Rs. 128 under the NWC method, and the cost of Rs. 156 under the LCE method. Hence, this method provides us with a better result that nearly approaches the optimal cost i.e. the minimum possible cost.

Thus, Vogel's approximation method helps us in obtaining almost the optimal solution with a fewer iterations.

5.15 OPTIMUM SOLUTION BY MODIFIED DISTRIBUTION (MODI) METHOD

Under this method evaluation is made of each of the empty cells (unoccupied cells) rather than the closed paths relating to each of the empty cells that is done under the stepping stone method. Here, we are to trace out only one closed path in relation to the cell that shows the highest negative net change in the cost of improvement. Thus, a considerable time is saved under this method.

The algorithm (procedure of working) is explained here as under :

Step 1. Find an initial basic feasible solution for the problem consisting of m + n — 1 stones under any of the three methods {viz ::NWCM, LCEM or VAM). Of these, however, Vogel's approximation method should be given preference in as much as its result approaches the optimal solution and thereby it needs few more iteration for getting the optimal solution under the MODI method.

Step 2. Determine the net change in the cost of each of the empty cells by the following equation.

Net change =C_{ij} — (R+ C)

Where C_{ij} = actual cost per unit shown in the concerned empty

cell of the ith row and jth column

(R + C) = sum of the implied cost of the concerned empty cell;

R = respective row as R_1, R_2 or R_3 of the concerned empty cell and

C = respective column as C_1, C_2 or C_3 of the concerned empty cell.

The value of the respective R and C of the empty cells are determined with reference to the value of the respective (R + C) of the stone square (stone cell). For this, the different rows of the Solution Table, are usually denoted as R_1, R_2, R_3 etc. starting from the top row and the different columns as C_1, C_2, and C_3 etc. starting from the left column.

Further the value of R_1 is to be taken as zero for finding the value of R_2, R_3, C_1, C_2 and C_3 etc. of the stone cells using the equation. $R+C_{ij}=C_j$.

Step 3: Evaluate the net change in the improvement cost of each of the empty cells. If, the net change in the improvement cost of all the empty cells is either positive or zero, the optimal solution is arrived at and accordingly the minimum cost will be determined as per the said optimum table. On the other hand, if any of the empty cells shows any negative value, it will indicate that the optimum solution is yet to be arrived at and accordingly the process of solution will be repeated further for optimizing the result.

For this, the next Solution Table will be revised as under:

For Revision of the Next Solution Table

1. Select the empty cell with the largest negative value as the Home square to be included in the next solution.

2. Draw a closed path (loop) in relation to this home square in the manner explained under the stepping stone method.
3. Put plus and minus signs alternatively in each of the cells at which the path turns in a right angle beginning with a plus sign in the home square. Skip over both the stones and the empty cells at which the path does not turn its direction.
4. Determine the smallest stone value associated with the minus sign and deduct the same from the stone values with the minus signs but add the same to the stone values with positive signs and the home square as well.

Step 4. After revision of the table in the above manner, repeat the step 2 and 3 explained above till an optimal solution is arrived at, where the net change in the improvement cost of all the empty cells are found to be either positive or zero.

Step 5. Determine the optimum cost at this level.

Note:

1. In case of maximization objective, all the indications like positive values of empty cells, plus and minus signs of the turning cells etc. will be just reversed.
2. At each stage of solution table, it must be seen that there is no degeneration, i.e. the number of stones does not fall short of m + n — 1. In case, this happens, dummy stones are to be introduced in any of the empty cells with the least possible cost to do away with such degenerating situation.

5.16 ASSGNMENT PROBLEM AND ITS SOLUTION:

The basic objective of an assignment problem is to assign n number of resources to n number of activities so as to minimize the total cost or to maximize the total profit of allocation in such a way that the measure of effectiveness is optimized. The problem of assignment arises because available resources such as men, machines, etc, have varying degree of efficiency for performing different activities such as job. Therefore cost, profit or time for performing the different activities is different. Hence, the problem is how should the assignments be made so as to optimize (maximize or minimize) the given objective. The assignment model can be applied in many decision making processes like determining optimum processing time in machine operators and jobs, effectiveness of teachers and subjects, designing of good plant layout, etc. This technique is found suitable for routing travelling salesmen to minimize the total travelling cost or to maximize the sales.

5.17 MATHEMATICAL STRUCTURE OF ASSIGNMENT PROBLEM

The structure of assignment problem of assigning operators to jobs is shown in table 5.1.

Table 5.1: Structure of Assignment Problem

Operator

	1	2		J		n
1	t_{11}	t_{12}	…..	t_{1j}	……	t_{1n}
2	t_{21}	t_{22}	…..	t_{2j}	…..	t_{2n}
I	t_{i1}	t_{i2}	….	t_{ij}	…..	t_{in}
n	t_{n1}	t_{n2}	….	t_{nj}	…..	t_{nn}

Let n be the number of jobs and number of operators.

Notes

t_{ij} be the processing time of job i taken by operator j.

A few applications of assignment problem are:

1. assignment of employees to machines.
2. assignment of operators to jobs.
3. effectiveness of teachers and subjects.
4. allocation of machines for optimum utilization of space.
5. salesmen to different sales areas.
6. clerks to various counters.

In all the cases, the objective is to minimize the total time and cost or otherwise maximize the sales and returns.

5.18 NETWORK REPRESENTATION OF ASSIGNMENT PROBLEM

An assignment model is represented by a network diagram in Figure 1 for an operator job assignment problem, given in Table 5.2 the time taken (in mins.) by operators to perform the job.

Table 5.2: Assignment Problem

Operator	Job		
	1	2	3
A	10	16	7
B	9	17	6
C	6	13	5

The assignment problem is a special case of transportation problem where all sources and demand are equal to 1.

5.19 USE OF LINEAR PROGRAMMING TO SOLVE ASSIGNMENT PROBLEM

A linear programming model can be used to solve the assignment problem. Consider the example shown in Table 5.2 to develop a linear programming model.

Let,

x_{11} represent the assignment of operator A to job 1

x_{12} represent the assignment of operator A to job 2

x_{13} represent the assignment of operator A to job 3

x_{21} represent the assignment of operator B to job 1

and so on

Formulating the equations for the time taken by each operator,

$10\ x_{11} + 16x_{12} + 7\ x_{13}$ = time taken by operator A.

$9\ x_{21} + 17x_{22} + 6x_{23}$ = time taken by operator B.

$6\ x_{31} + 13x_{32} + 5x_{33}$ = time taken by operator C.

The constraint in this assignment problem is that each operator must be assigned to only one job and similarly, each job must be performed by only, one operator. Taking-this constraint into account, the constraint equations are as follows:

Notes

$x_{11} + x_{12} + x_{13} \leq 1$ operator A

$x_{21} + x_{22} + x_{23} \leq 1$ operator B

$x_{31} + x_{32} + x_{33} < 1$ operator C

$x_{11} + x_{21} + x_{31} = 1$ Job 1

$x_{12} + x_{22} + x_{32} = 1$ Job 2

$x_{13} + x_{23} + x_{33} = 1$ Job 3

Objective function: The objective function is to minimize the time taken to complete all the jobs. Using the cost data table; the following equation can be arrived at:

The objective function is,

Minimize $Z = 10x_{11}+16\ x_{12}+7x_{13}+9x_{21}+17x_{22}+\ 6x_{23}+6x_{31}+13x_{32}+5x_{33}$

The linear programming model for the problem will be,

Minimize $Z = 10x_{11}+16x_{12}+7x_{13}+9x_{21}+17x_{23}+6x_{23}+6x_{31}+13x_{32}+5x_{33}$

subject to constraints

$x_{11} + x_{12} + x_{13} \leq 1$ ………………(i)

$x_{21} + x_{22} +.x_{23} \leq 1$ ……………..(ii)

$x_{31}+ x_{32}+ x_{33} \leq l$ ……..……….(iii)

$x_{11} + x_{12} + x_{13} = 1$ ………… …(iv)

$x_{12} + x_{22} + x_{23} = 1$ ……………..(v)

$x_{13}+ x_{23}+ x_{33} = l$ ……..……….(vi)

where, $x_{ij} \geq 0$ for i=1,2,3 and j=1,2,3

5.20 TYPES OF ASSIGNMENT PROBLEM

The assignment problems are of two types (i) balanced and (ii) unbalanced. If the number of rows is equal to the number of columns or if the given problem is a square matrix, the problem is termed as a balanced assignment problem. If the given problem is not a square matrix, the problem is the termed as an unbalanced assignment problem.

If the problem is an unbalanced one, add dummy rows /dummy columns as required so that the matrix becomes a square matrix or a balanced one. The cost or time values for the dummy cells are assumed as zero.

5.21 HUNGARIAN METHOD FOR SOLVING ASSIGNMENT PROBLEM

Step 1: In a given problem, if the number of rows is not equal to the number of columns and vice versa, then add a dummy row or a dummy column. The assignment costs for dummy cells are always assigned as zero.

Step 2: Reduce the matrix by selecting the smallest element in each row and subtract with other elements in that row.

Step 3: Reduce the new matrix column-wise using the same method as given in step 2.

Step 4: Draw minimum number of lines to cover all zeros.

Step 5: If Number of lines drawn = order of matrix, then optimally is reached, so proceed to step 7. If optimally is not reached, then go to step 6.

Notes

Step 6:- Select the smallest element of the whole matrix, which is NOT COVERED by lines. Subtract this smallest element with all other remaining elements that are NOT COVERED by lines and add the element at the intersection of lines. Leave the elements covered by single line as it is. Now go to step 4.

Step 7: Take any row or column which has a single zero and assign by squaring it. Strike off the remaining zeros, if any, in that row and column (X). Repeat the process until all the assignments have been made.

Step 8: Write down, the assignment results and find the minimum cost/time.

Note: While assigning, if there is no single zero exists in the row or column, choose any one zero and assign it. Strike off the remaining zeros in that column or row, and repeat the same for other assignments also. If there is no single zero allocation, it means multiple number of solutions exist But the cost will remain the same for different sets of allocations.

5.22 SUMMARY

Linear programming is a mathematical technique which is applied in the form of a linear formula for arriving at a rational proportion of the variables to be used as inputs to get the optimum result from a course of action to be planned accordingly.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying model.

The assignment model can be applied in many decision-making processes like determining optimum processing time in machine operators and jobs, effectiveness of teachers and subjects, designing of good plant layout, etc. This technique is found suitable for routing travelling salesman to minimize the total travelling cost, or to maximize the sales.

5.23 SELF ASSESSMENT QUESTIONS

1. Define a linear programming problem. State the different types of linear programming problems and briefly point out the limitations of an L.P.P.
2. Explain briefly the formulation procedure of a linear programming problem.
3. What are the three major problems that can be solved using the linear programming techniques? Discuss each of them briefly.
4. Explain briefly the following concepts of linear programming problem.

   a) Objective function

   b) Constraint function

   c) Non-negativity conditions
5. From the following data formulate the L.P. model for the A company Ltd.

Product	**Work Centre**			**Profit**
	A	B	C	
P	1 hrs.	2 hrs	3 hrs.	80
G	3 hrs.	5 hrs.	1 hr.	100
Total Capacity	720 hrs.	1800 hrs	900 hrs.	

6. A firm produces two types of products P and Q and sells them at a profit of Rs.2 and Rs.3 respectively. Each product passes through two machines, R and S. P requires one minute of processing time on R and two minutes on S. Q requires one minute on R andone minute on S.

Notes

The machine R is available for not more than 6 hours and 40 minutes while the machine S is available for 10 hours during any working day.

Formulate the above as a linear programming problem.

7. A firm produces two types of mats. Each mat of the first type needs twice as much labour as the second type. If all the mats are of the second type only, the firm can produce a total of 500 unit mats a day. The market limits daily sales of the first and second type to 150 and 250 respectively. Assuming that the profit per mat are Rs.8 and Rs.5 respectively for the two types, formulate the problem as a linear programming model to determine the number of mats to be produced of each type so as to maximize the profit.

8. A company produces three products, X, Y and Z. The profits are at the rate of Rs.3, Rs. 2 and Rs. 4 respectively. The company has two machines P and Q. Below are given the processing times in hours for each machines on each product.

Machine	Product		
	X	Y	Z
P	4	3	5
Q	3	2	4

Machines P and Q have 2000 and 2500 machines hours respectively. The company shall produce 100X, 200Y and 50Z but not more than 150X. Set up an L.P model to maximize the profit.

9. What do you understand by a transportation problem? State the essential characteristics of such a problem.

10. Explain in brief the various methods of obtaining an initial feasible solution to a transportation problem.

11. Write Notes on

a. North West Corner rule

b. Lowest Cost Entry method

c. Vogel’s Approximation method

d. Modified distribution (MODI) method.

12. Explain the difference between a transportation problem and an assignment problem. Explain the situations where a transportation problem can arise.

13. Solve the following L.P.P by graphic method:

Maximize $Z = 5x_1 + 3x_2$

Subject to $3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$